GEVREY-TYPE REGULARITIES AND SHIFT-INVARIANCE IN SOBOLEV-TYPE SPACES: MIDTERM RESULTS OF THE GOALS PROJECT

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Abstract. We present a brief overview of the results obtained during the first two years of the GOALS project. The first group of results concerns Gevrey regularity on positive orthants. We then review several aspects of the so-called extended Gevrey regularity. Finally, we investigate various properties of shift-invariant subspaces of Sobolev type.

0. Introduction

Project GOALS was launched in December 2023 with the aim of bringing together researchers working on topics related to spaces of test functions and to perform a detailed analysis of different regularity properties. At a more advanced level, the project team also aims to study operators acting on such test functions and their distributions. To ensure a multidisciplinary approach and explore the topics from various perspectives, the work plan is organized in four work packages, three of which were completed within the first 20 months of the project. This note provides a brief overview of the project deliverables achieved during that period and beyond.

Through the dedication and synergy of the team members, we succeeded in producing more articles than initially planned in the project proposal. Additionally, several new and promising research topics were identified, which will be investigated during the final year of the project. To enhance the overall research output and open opportunities for further investigations after the project ends, we recommend a slight variation of the originally planned contents of WP4. This will allow us to achieve scientific results and outputs that can be regarded as an important added value of the GOALS project.

1. WP1 –Ultradistributions via time–frequency analysis

WP1 team members (S. Jakšić, N. Teofanov, F. Tomić, Đ. Vučković) studied specific spaces of test functions and their distribution spaces. Several investigations were focused on spaces over positive orthants. Apart from structural properties of the spaces, various integral transforms acting on them were considered.

1.1. **Ultradifferentiable functions and iterates of the Laguerre operator.** Pilipović spaces over $\mathbb{R}^d$ are known to be an extension of the concept of Fourier transform invariant Gelfand-Shilov type spaces. Their counterparts on $\mathbb{R}^d_+$ have not been investigated so far. In [10] we announce investigations in this direction.

More precisely, we use iterates of the Laguerre operator to introduce Pilipović spaces on positive orthants. It is shown that such spaces coincide with $G-$type spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$, when $\alpha > 1$, and $\alpha \geq 1$, respectively. However, in contrast to G-type spaces, Pilipović spaces are nontrivial below the critical index $\alpha = 1$. We also investigate a natural isomorphism between subspaces of Pilipović spaces on $\mathbb{R}^d$ consisting of even functions, and Pilipović spaces on positive orthants.

The results announced in [10] are developed in full detail in [11].

We define and characterize ultradifferentiable functions and their corresponding ultradistributions on $\mathbb{R}^d_+$ using iterates of the Laguerre operator. The characterization is based on the decay or growth conditions of the coefficients in their Laguerre series expansion. We apply our results to establish an isomorphism between subspaces of Pilipović spaces on $\mathbb{R}^d$, and the spaces of ultradifferentiable functions on $\mathbb{R}^d_+$.

1.2. **Fractional Hankel Transform of generalized functions.** Abelian and Tauberian type results are central results when considering asymptotic behavior of integral tranforms. According to P. Duren, "An Abelian theorem is any statement to the effect that a method of summability assigns to each convergent series its ordinary sum. A Tauberian theorem goes in the opposite direction and asserts that every summable series which is not too badly divergent is actually convergent."[1] A similar situation arises when summations are replaced by integrations, in which case a "method of summability" corresponds to a specific integral transform.

In [7] we explore the quasiasymptotic behavior of distributions through the fractional Hankel transform. We present Tauberian result that connects the asymptotic behavior of generalized functions in the Zemanian space with the asymptotics of their fractional Hankel transform. Additionally, we establish both the initial and final value theorems for the fractional Hankel transform of distributions.

The paper [8] can be considered as a natural continuation of [7]. More precisely, in [8] we first present an Abelian-type theorem for the fractional Hankel transform (FrHT) within Zemanian generalized function spaces. To prove this, we show that these spaces have the Montel property. Next, we construct a new Zemanian-type space as a projective limit of suitable Banach spaces. Its dual is the largest known distribution space admitting the FrHT. Finally, within this extended setting, we establish new Abelian and Tauberian-type results for the FrHT.

1.3. **Ultradistributions on $\mathbb{R}^d_+$, hypoellipticity and the Weyl Calculus of Pseudo-Differential Operators.** Another line of investigations related to the spaces on positive orthants is pushed forward by considering different types of operators acting on such spaces.

In [17] we first analyze spaces $\mathcal{G}^*(\mathbb{R}^d_+)$ and their dual spaces of ultradistributions through Laguerre expansions. These spaces correspond to a general sequence $\{M_p\}_{p\in\mathbb{N}_0}$, where * is a common notation for the Beurling and Roumieu classes. Recall that in the Beurling setting one considers spaces formed as inductive limits with respect to an auxiliary parameter, while in the Roumieu setting the construction is based on projective limits.

[1] P. Duren, Invitation to Classical Analysis, AMS, 2012

In the second part of [17] we are solving equations of the form

$$Lu = f, \qquad L = \sum_{j=1}^{k} a_j A_j^{h_j} + cE_y^d + bP(x, D_x),$$

where f belongs to the tensor product of ultradistribution spaces over compact manifolds without boundaries as well as ultradistribution spaces on $\mathbb{R}^n_+$ and $\mathbb{R}^d$, $A_j, j = 1, ..., k$, E_y and $P(x, D_x)$ are operators whose eigenfunctions form orthonormal basis of corresponding L^2-space. The sequence space representation of solutions enable us to study the solvability and the hypoellipticity in the specified spaces of ultradistributions.

In [9] we introduce classes of ultradifferentiable functions and corresponding ultradistributions on $\mathbb{R}^d_+$, i.e. $G^\beta_\alpha(\mathbb{R}^d_+)$ and $(G^\beta_\alpha(\mathbb{R}^d_+))'$, $\alpha, \beta > 0$, respectively. We give their characterisation through the Laguerre coefficients estimate. Furthermore, we define the modified fractional power of the partial Hankel-Clifford transform and show that this transformation is a topological isomorphism on $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$. We also investigate the boundedness of the Weyl pseudo-differential operators with symbols from the previously introduced spaces.

1.4. **Wiener–Lebesgue type spaces.** One of the objectives of the GOALS project is to investigate situations in which local properties are combined or amalgamated with global ones. This perspective can be traced back to N. Wiener and his theory of generalized harmonic analysis.[2]

In [14] we investigate the relationship between quasi-Banach modulation spaces and Wiener-Lebesgue amalgam spaces. More precisely, we establish norm equivalence for these spaces across the full range of the Lebesgue parameters. Our main result unifies and complements the findings known so far.

In [6] we extend dilation properties of Wiener amalgam spaces when the local and global componenets are Lebesgue spaces to a more general setting of Orlicz spaces. We recover the result of Cordero and Nicola when restricted to Lebesgue spaces. In addition, we prove continuity of the Zak transform on Wiener amalgam spaces with Orlicz spaces as their local components.

2. WP2 – Pseudo-differential operators and wave–front sets

To provide a suitable framework for the study of pseudo-differential operators, the WP2 team members (F. Tomić, S. Aleksić, S. Jakšić, N. Teofanov, S. Tutić, M. Žigić) investigated various properties of test function spaces associated with specific regularity conditions. This (extended Gevrey) regularity is weaker than that in Gevrey spaces but stronger than mere smoothness. These investigations led to several notable results, including the construction of new orthonormal wavelet bases.

[2]N. Wiener, Tauberian theorems, Ann. of Math. **33** (1932) 1–100.

2.1. **Almost diagonalization of ΨDO's over various generalized function spaces.** In [16] we survey some recent results related to the operators of symbol–global type. Inductive and projective type sequence spaces of sub- and super-exponential growth, and the corresponding inductive and projective limits of modulation spaces are considered as a framework for almost diagonalization of pseudo-differential operators. Moreover, recent results of the first author and B. Prangoski related to the almost diagonalization of pseudo-differential operators in the context of Hörmander metrics are reviewed.

2.2. **Band-limited wavelets beyond Gevrey regularity.** In [18] we give a construction of orthonormal wavelets with specific regularity properties.

It is known that a smooth function of exponential decay at infinity can not be an orthonormal wavelet. Dziubański and Hernández constructed smooth orthonormal wavelets of Gevrey type subexponential decay. We weaken the Gevrey type decay and construct orthonormal wavelets of subexponential decay related to the so-called extended Gevrey classes. The virtue of our construction is that prescise asymptotics of functions from such classes can be given in terms of the Lambert W function.

2.3. **An introduction to extended Gevrey regularity.** In [19] we present an overview of recent results related to extended Gevrey regularity.

Gevrey classes are the most common choice when considering the regularities of smooth functions that are not analytic. However, in various situations, it is important to consider smoothness properties that go beyond Gevrey regularity, for example when initial value problems are ill-posed in Gevrey settings. In this paper we consider a convenient framework for studying smooth functions that possess weaker regularity than any Gevrey function. Since the available literature on this topic is scattered, our aim is to provide an overview to extended Gevrey regularity, highlighting its most important features. Additionally, we consider related dual spaces of ultradistributions and review some results on micro-local analysis in the context of extended Gevrey regularity. We conclude the paper with a few selected applications that may motivate further study of the topic.

2.4. **Gelfand-Shilov Spaces for Extended Gevrey Regularity.** In [20] we consider spaces of smooth functions obtained by relaxing Gevrey-type regularity and decay conditions. It is shown that these classes can be introduced by using the general framework of the weighted matrices approach to ultradifferentiable functions. We examine alternative descriptions of Gelfand-Shilov spaces related to the extended Gevrey regularity and derive their nuclearity. In addition to the Fourier transform invariance property, we present their corresponding symmetric characterizations. Finally, we consider some time-frequency representations of the introduced classes of ultradifferentiable functions.

3. WP3 – Frames for ultradistributions

The main objective of WP3 (team members: S. Aleksić, A. Aksentijević, N. Teofanov, Đ. Vučković, M. Žigić) is to provide a detailed study of shift-invariant spaces using frame theory. Additionally, to investigate nonlinear phenomena and shift-invariance in a broader

context, the Colombeau theory of generalized functions and the theory of semigroups are employed, leading to several noteworthy additional deliverables.

3.1. **Shift-invariant subspaces of Sobolev type.** The paper [2] has the characteristics of a review paper in which results of shift-invariant subspaces of Sobolev type are summarized without proofs. The structure of shift-invariant spaces V_s, $s \in \mathbb{R}$, generated by at most countable family of generators, which are subspaces of Sobolev spaces $H^s(\mathbb{R}^d)$, are announced in [3] and Bessel sequences, frames and Riesz families of such spaces are characterized. With the Fourier multiplier

$$\left(1 - \frac{\Delta}{4\pi^2}\right)^{s/2} f = \mathcal{F}^{-1}\big((1+|t|^2)^{s/2}\widehat{f}(t)\big),$$

we are able to extend some well known notions and theorems to spaces of the Sobolev type.

In [3] product in shift-invariant spaces, and periodic distributions are investigated.

We connect through the Fourier transform shift-invariant Sobolev type spaces $V_s \subset H^s$, $s \in \mathbb{R}$, and the spaces of periodic distributions and analyze the properties of elements in such spaces with respect to the product. If the series expansions of two periodic distributions have compatible coefficient estimates, then their product is a periodic tempered distribution. We connect product of tempered distributions with the product of shift-invariant elements of V_s. The idea for the analysis of products comes from the Hörmander's description of the Sobolev type wave front in connection with the product of distributions. Coefficient compatibility for the product of f and g in the case of "good" position of their Sobolev type wave fronts is proved in the 2-dimensional case. For larger dimension it is an open problem because of the difficulties on the description of the intersection of cones in dimension $d \geq 3$.

The results of [3] are complemented by the findings presented in [4], where we study the shift-preserving operator $L : V_s \to V_s$ and the range operator R_s and their relationship. Here V_s denotes a shift-invariant subspace of Sobolev space $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$. Using the range operator, we give a result about dual frames. For the shift-invariant space V_s generated by n functions, we find conditions on L and a finite set $\{\phi_i : \phi_i \in V_s, i = 1, \ldots, m\}$ so that the collection $\{L^j\phi_i : i = 1, \ldots, m, j = 0, \ldots, n-1\}$ is a frame generator for finitely generated shift-invariant space V_s.

In [5] we discuss some structural properties of finitely generated shift-invariant (FGSI) spaces and subspaces of Sobolev spaces, particularly those related to convolution and the product within these spaces. We find shift-invariant solutions in FGSI spaces for a class of differential-difference equations with constant coefficients. Additionally, we analyze the Fourier multipliers in FGSI spaces and the wave fronts for the convolution and product in FGSI spaces.

Most of the results mentioned in 3.1 are contained in A. Aksentijević's PhD thesis [1].

3.2. **Topology and regularity for generalized ultradistribution algebras.** Several extensions of the classical Schwartz theory of distributions are known, among which the Colombeau theory serves as a prominent example. In [15], one of the established results

concerns shift-invariant generalized ultradistributions and emerges as a consequence of the underlying topological structure of the spaces considered.

Compiling essential results for non-quasianalytic ultradistribution spaces and Colombeau versions of generalized ultradistribution algebras, in [15] analyze strong B- and strong R-association of a generalized ultradistribution $[(f_\varepsilon)]$. The strong association of $[(f_\varepsilon)]$ with a Komatsu-type ultradistribution T, with an additional assumption on regularity of $[(f_\varepsilon)]$ of Beurling, respectively, Roumieu type, implies that T is an ultradifferentiable function of Beurling, Roumieu type, respectively. We show that under suitable conditions, a weakly negligible net $(f_\varepsilon)_{\varepsilon\in(0,1)}$ is a negligible net in the sense of generalized ultradistributions. Furthermore, we prove that a translation-invariant generalized ultradistribution g is equal to a generalized constant in both types of generalized ultradistribution algebras.

3.3. **Generalized exponentially bounded integrated semigroups and C_0-semigroups.** An application-oriented facet of WP3 is devoted to the analysis and characterization of solutions to specific partial differential equations.

The subject of [12] is the analysis of sequences of infinitesimal generators and exponentially bounded integrated semigroups which are related to Cauchy problems

$$\frac{\partial}{\partial t}u(t,x) - a(D)u(t,x) = f(t,x), \quad u(0,x) = u_0(x), \quad t \geq 0,\ x \in \mathbb{R}^d, \tag{1}$$

with distributional initial data u_0 and distributional right hand sides f through sequences of equations with regularized u_0 and f, and $a(D)$ approximated by a suitable sequences of (pseudo)differential operators $a_n(D)$. Mainly, the paper deals with the comparison of sequences of infinitesimal generators and the determination of corresponding sequences of integrated semigroups. For this purpose, we introduce association, the relation of equivalence for infinitesimal generators on one side and the corresponding relations of equivalence of integrated semigroups on another side. The order of involved assumptions on generators essentially characterize the mutual dependence of sequences of infinitesimal generators and the corresponding sequences of integrated semigroups. Our motivation is presented with an example showing how Cauchy problem (1) with certain type of singularities is transferred to a sequence of Cauchy problems with classical solutions presented by one time integrated semigroups, and then discuss the solution to (1) in a weak sense (in the sense of distributions) or a very weak sense (when only a sequence of solutions exists without a weak limit). The effects of perturbations of infinitesimal generators to integrated semigroups is the core of the paper.

Furthermore, in [13] we study nonlinear stochastic partial differential equations with Wick-analytic type nonlinearities set in the framework of white noise analysis. These equations include the stochastic Fisher–KPP equations, stochastic Allen–Cahn, stochastic Newell–Whitehead–Segel, and stochastic Fujita–Gelfand equations. By implementing the theory of C_0-semigroups and evolution systems into the chaos expansion theory in infinite dimensional spaces, we prove existence and uniqueness of solutions for this class of stochastic partial differential equations.

4. WP4 – Local vs. global descriptions

The lengthy, paper [21] serves as an excellent foundation for further research, in which the interplay between local and global properties plays a decisive role. All team members will be involved in the execution of WP4.

More precisely, in [21] we consider a broad class of modulation spaces $M(\omega, \mathscr{B})$, parameterized with weight function ω and a normal quasi-Banach function space $\mathscr{B}$ of order $r_0 \in (0, 1]$. Then we prove that $f \in M(\omega, \mathscr{B})$, if and only if $V_\phi f$ belongs to the Wiener amalgam space $\mathsf{W}^r(\omega, \mathscr{B})$, and

$$\|f\|_{M(\omega,\mathscr{B})} \asymp \|V_\phi f \cdot \omega\|_{\mathscr{B}} \asymp \|V_\phi f\|_{\mathsf{W}^r(\omega,\mathscr{B})}, \quad r \in [r_0, \infty].$$

We use the results to extend and improve continuity and lifting properties for pseudo-differential and Toeplitz operators with symbols in weighted M^{∞,r_0}-spaces, $r_0 \le 1$, when acting on $M(\omega, \mathscr{B})$-spaces.

5. Conclusion

The results obtained during the first two years of the GOALS project indicate that team members have developed the skills and expertise required to deliver outcomes of high scientific quality. We produced more deliverables than initially expected and conducted investigations across a broader set of research fields than originally planned.

Our investigations have opened new opportunities to deepen the understanding of the considered phenomena. We aim to continue these explorations during the implementation of WP4. This will lead to a slight deviation from some of the topics outlined in the project proposal. Such adjustments are expected to enhance the overall research output beyond the originally anticipated level.

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