Shift-invariant subspaces of Sobolev spaces and shift-preserving operators

Aleksandar Aksentijević, Suzana Aleksić and Stevan Pilipović

Abstract. We study the shift-preserving operator $L : V_s \to V_s$ and the range operator R_s and their relationship, where V_s is a shift-invariant subspace of Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Using the range operator, we give a result about dual frames. For the shift-invariant space V_s generated by d functions, we find conditions on L and a finite set $\{\phi_i : \phi_i \in V_s, i = 1, \ldots, m\}$ so that the collection $\{L^j \phi_i : i = 1, \ldots, m, j = 0, \ldots, d-1\}$ is a frame generator for finitely generated shift-invariant space V_s.

Key Words and Phrases: Sobolev space; shift-invariant space; range function; range operator; shift-preserving operator; frame; dual frame, dimension function, dynamical sampling, diagonalization.

2020 Mathematics Subject Classification: 42C40, 42C15, 42B30, 46B15, 42B30, 47A05, 47A15, 94A20.

1. Introduction

This paper has the characteristics of a review paper in which results of Bownik [17] and Aguilera et al. [5]-[6] were transferred to Sobolev spaces, thus it also contains the original results. If after some assertion is written ([...], Theorem/Lemma/Proposition ...), it means that the corresponding result is proved in quoted paper for the L^2-case.

In the first part of this paper, we follow the results of Bownik [17] (based on [15]-[16], [24], [27]) for shift-preserving operator defined on shift-invariant subspaces of $L^2(\mathbb{R}^n)$, and give the corresponding results for the shift-preserving operator defined on V_s, where V_s is a shift-invariant subspace of Sobolev space $H^s := H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, building on our results [7]. By studying the shift-preserving operator defined on V_s, we give some characteristics of the dimension function. We show that every shift-preserving operator $L : V_s \to H^s$ can be represented by the corresponding range operator R_s.

Using a range operator approach, for a given frame with bounds A and B, we obtain elements of the dual frame with bounds B^{-1} and A^{-1}.

In the second part of this paper, we follow the results of [6] (based on [5], [8], [17]-[18]) and thus, the definitions of s-eigenvalue, s-eigenspace and s-diagonalization of the shift-preserving operator L are adapted to our spaces. We prove that every normal shift-preserving operator $L : V_s \to V_s$ is s-diagonalizable. Moreover, if the operator L is s-diagonalizable then the corresponding range operator is diagonalizable. Finally, we give conditions under which the collection $\{L^j\phi_i : \phi_i \in V_s, i = 1,\ldots,m, j = 0,\ldots,d-1\}$ is a frame generator for shift-invariant space V_s generated by d functions $\varphi_1,\ldots,\varphi_d \in H^s$. This problem is known as Dynamical Sampling. The dynamical sampling problem has got a lot of attention in the last few years. For a recent account of the theory, we refer the reader to [1]-[3], [8]-[11], [20]-[22], [25]. Applications can be found in [12], [29].

The paper is organized as follows. After the introduction section where background information is provided, in Section 2 we reintroduce notation and assertions in much the same way as in [7]. The shift-preserving operator and the range operator are studied in Section 3. In Section 4, we prove that the frame operator has the corresponding range operator associated with the dual Gramian and we give a result about the dual frame. The results in Section 6 will prove extremely useful in Section 7. Finally, in Section 7, we find conditions on the shift-preserving operator and a finite set of functions of V_s so that the iterations of the operator on the functions produce a frame generator set of V_s.

2. Notation and basic assertions

We follow the notation of [7]. Let s be a real number, $\mu_s(\cdot) := (1 + |\cdot|^2)^{s/2}$ and $\mathbb{T}^n := [0,1)^n$, where $n \in \mathbb{N}$. We denote the translation of a function f for $k \in \mathbb{Z}^n$ by $T_k f(x) = f(x-k)$. By $\widehat{f}$ or $\mathcal{F}f$ we denote the Fourier transform of an integrable function f defined by $\mathcal{F}f(t) = \widehat{f}(t) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi\sqrt{-1}\langle x,t\rangle}\,\mathrm{d}x$, $t \in \mathbb{R}^n$, where $\langle x,t\rangle = \sum_{i=1}^n x_i t_i$. Note that $\mathcal{F}^{-1}f(t) = \widehat{f}(-t)$. Recall,

$$(c_k)_{k\in\mathbb{Z}^n} \in \ell^2_s(\mathbb{Z}^n) \quad \text{if and only if} \quad \sum_{k\in\mathbb{Z}^n} |c_k|^2\mu_s^2(k) < +\infty,$$

$$f \in H^s \quad \text{if and only if} \quad f \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\widehat{f}(t)|^2\mu_s^2(t)\,\mathrm{d}t < +\infty,$$

with the scalar product $\langle (c_k)_{k\in\mathbb{Z}^n}, (d_k)_{k\in\mathbb{Z}^n}\rangle_{\ell^2_s} = \sum_{k\in\mathbb{Z}^n} c_k\overline{d}_k\mu_s^2(k)$ for $\ell^2_s(\mathbb{Z}^n)$ and $\langle f,g\rangle_{H^s} = \int_{\mathbb{R}^n}\widehat{f}(t)\overline{\widehat{g}}(t)\mu_s^2(t)\,\mathrm{d}t$ for H^s. To shorten notation, we write ℓ^2_s instead of $\ell^2_s(\mathbb{Z}^n)$. Note, $L^2_s := L^2_s(\mathbb{R}^n) = \mathcal{F}(H^s)$, i.e. $f \in L^2_s$ if and only if $\widehat{f} \in H^s$, $s \in \mathbb{R}$. By $H\left(\mathbb{T}^n, \ell^2_s\right)$ we denoted the Hilbert space which consists of all vector-valued measurable functions $F : \mathbb{T}^n \to \ell^2_s$ such that $\int_{\mathbb{T}^n}\|F(t)\|^2_{\ell^2_s}\,\mathrm{d}t < +\infty$, with scalar product $\langle F_1,F_2\rangle_{H(\mathbb{T}^n,\ell^2_s)} = \int_{\mathbb{T}^n}\langle F_1(t),F_2(t)\rangle_{\ell^2_s}\,\mathrm{d}t$. By I is denoted the finite set $\{1,2,\ldots,r\}$ or $\mathbb{N}$. Next, $E_s(\mathcal{A}_{I,s}) = \{T_k\varphi : \varphi \in \mathcal{A}_{I,s}, k \in$

$\mathbb{Z}^n\} \subset H^s$, where $\mathcal{A}_{I,s} = \{\varphi_i : \varphi_i \in H^s, i \in I\}$. If $I = \{1,\dots,r\}$, we will use notation $\mathcal{A}_{r,s}$ instead of notation $\mathcal{A}_{I,s}$.

Recall ([7], [17]), a closed subspace $V_s \subset H^s$ is a shift-invariant space if $T_k\varphi \in V_s$ for every $\varphi \in V_s$, for any $k \in \mathbb{Z}^n$. The space $S_s(\mathcal{A}_{I,s}) = \overline{\operatorname{span}}(E_s(\mathcal{A}_{I,s}))$ is a shift-invariant space generated by $\mathcal{A}_{I,s}$. It is called a principal shift-invariant (PSI) space if $S_s(\mathcal{A}_{I,s}) = S_s(\varphi)$. If $S_s(\mathcal{A}_{I,s}) = S_s(\varphi_1, \varphi_2, \dots, \varphi_r)$, we say that it is a finitely generated shift-invariant (FSI) space.

We recall some assertions from [7].

Lemma 2.1 ([7]). *A mapping $\mathcal{T}_s : H^s \to H(\mathbb{T}^n, \ell^2_s)$ defined by*

$$\mathcal{T}_s\varphi(t) = \left(\frac{\widehat{\psi}(t+k)}{\mu_s(k)}\right)_{k\in\mathbb{Z}^n}, \quad t \in \mathbb{T}^n,$$

where $\varphi \in H^s$ and $\left(1-\frac{\Delta}{4\pi^2}\right)^{s/2}\varphi = \psi \in L^2(\mathbb{R}^n)$, is an isometric isomorphism and

$$\mathcal{T}_s T_j\varphi(t) = \mathrm{e}^{-2\pi\sqrt{-1}\langle t,j\rangle}\mathcal{T}_s\varphi(t), \quad j \in \mathbb{Z}^n, t \in \mathbb{T}^n.$$

We introduce several notions, following [7]. A range function is a mapping

$$J_s : \mathbb{T}^n \to \left\{\text{closed subspaces of } \ell^2_s\right\}.$$

Next, J_s is measurable if for each $a, b \in \ell^2_s$, the scalar function $t \mapsto \langle P_{J_s}(t)a, b\rangle_{\ell^2_s}$ is measurable, where $P_{J_s}(t) : \ell^2_s \to J_s(t)$, $t \in \mathbb{T}^n$, are associated orthogonal projections. For a function J_s, let us define the space $M_{J_s} = \{F \in H(\mathbb{T}^n, \ell^2_s) : F(t) \in J_s(t) \text{ for a.e. } t \in \mathbb{T}^n\}$, which is a closed subspace of $H(\mathbb{T}^n, \ell^2_s)$.

Theorem 2.1 ([7]). *Let J_s be a measurable range function. A closed subspace $V_s \subset H^s$ is shift-invariant if and only if*

$$V_s = \left\{\varphi \in H^s \,:\, \mathcal{T}_s\varphi(t) \in J_s(t) \text{ for a.e. } t \in \mathbb{T}^n\right\}.$$

The correspondence between V_s and J_s is one-to-one under the convention that the range functions are identified if they are equal a.e. Furthermore, if $V_s = S_s(\mathcal{A}_{I,s})$ for some $\mathcal{A}_{I,s} \subset H^s$, then $J_s(t) = \overline{\operatorname{span}}\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\}$.

Thus, every measurable range function J_s determines a shift-invariant space V_s by $V_s = \mathcal{T}_s^{-1}M_{J_s}$, and vice versa, i.e. for every shift-invariant space $V_s \subset H^s$ there is a measurable range function J_s such that $\mathcal{T}_sV_s = M_{J_s}$. The definition of the dimension function $\dim_{V_s} : \mathbb{T}^n \to \mathbb{N} \cup \{0, +\infty\}$ of V_s is adapted from [7] and it is defined by $\dim_{V_s}(t) = \dim J_s(t)$, where $V_s = \mathcal{T}_s^{-1}M_{J_s}$. The spectrum of V_s is defined by $\sigma_{V_s} := \left\{t \in \mathbb{T}^n : J_s(t) \neq \{\mathbf{0}\}\right\}$.

Recall that $E_s(\mathcal{A}_{I,s})$ is a Bessel family for $S_s(\mathcal{A}_{I,s})$ with bound $B < +\infty$ if

$$\sum_{(k,i)\in\mathbb{Z}^n\times I} |\langle f, T_k\varphi_i\rangle_{H^s}|^2 \le B\|f\|^2_{H^s}, \qquad f \in \operatorname{span}(E_s(\mathcal{A}_{I,s})).$$

A Bessel family $E_s(\mathcal{A}_{I,s})$ is called a frame for $S_s(\mathcal{A}_{I,s})$ with frame bounds A, B, if additionally there exists $A > 0$ such that

$$A\|f\|^2_{H^s} \le \sum_{(k,i)\in\mathbb{Z}^n\times I} |\langle f, T_k\varphi_i\rangle_{H^s}|^2, \qquad f \in \operatorname{span}(E_s(\mathcal{A}_{I,s})).$$

We call $E_s(\mathcal{A}_{I,s}) \subset H^s$ a Riesz family with bounds $A, B > 0$ if

$$A\|c\|^2_{\ell^2_s} \le \Big\| \sum_{(k,i)\in\mathbb{Z}^n\times I} c_i T_k\varphi_i \Big\|^2_{H^s} \le B\|c\|^2_{\ell^2_s},$$

hold for all (finitely supported) sequences $c = (c_i)_{i\in I}$. The best general references here are [19] and [23].

Theorem 2.2 ([7]). *$E_s(\mathcal{A}_{I,s})$ is a frame or a Riesz basis for $S_s(\mathcal{A}_{I,s})$ with bounds A, B or a Bessel family with bound B for every $s \in \mathbb{R}$ (equivalently, for some $s \in \mathbb{R}$) if and only if $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \subset \ell^2_s$ is a frame or a Riesz basis for $J_s(t)$ with bounds A, B or a Bessel family with bound B for a.e. $t \in \mathbb{T}^n$, for every $s \in \mathbb{R}$ (equivalently for some $s \in \mathbb{R}$), respectively. Moreover, $E_s(\mathcal{A}_{I,s})$ is a fundamental frame for every $s \in \mathbb{R}$ if and only if $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \subset \ell^2_s$ is a fundamental frame for a.e. $t \in \mathbb{T}^n$, for every $s \in \mathbb{R}$.*

We say that $\varphi_0 \in S_s(\varphi)$ is a tight-frame generator (or quasi-orthogonal generator) of $S_s(\varphi)$ if for every $f \in S_s(\varphi)$ holds $\|f\|^2_{H^s} = \sum_{k\in\mathbb{Z}^n} |\langle T_k\varphi_0, f\rangle_{H^s}|^2$. If φ_0 is a tight-frame generator of $S_s(\varphi)$, then $\|\mathcal{T}_s\varphi_0(t)\|_{\ell^2_s} = \mathbf{1}_{\sigma_{S_s(\varphi)}}(t)$ for a.e. $t \in \mathbb{T}^n$, and vice versa (see [7] for more details).

The next theorem is given in [7] without proof, since it is similar to the proof of Theorem 3.3 in [17]. For the sake of completeness, we give here the proof.

Theorem 2.3 ([7]). (*The decomposition theorem*). *Suppose that V_s is a shift-invariant subspace of H^s. Then, V_s can be decomposed as an orthogonal sum*

$$V_s = \bigoplus_{i\in\mathbb{N}} V_s^i, \tag{2.1}$$

where V_s^i, $i \in \mathbb{N}$, are principal shift-invariant spaces with tight-frame generators φ_i, $i \in \mathbb{N}$, and $\sigma_{V_s^{i+1}} \subset \sigma_{V_s^i}$, for all $i \in \mathbb{N}$. Moreover, $\dim_{V_s^i}(t) = \|\mathcal{T}_s\varphi_i(t)\|_{\ell^2_s}$, $i \in \mathbb{N}$, and

$$\dim_{V_s}(t) = \sum_{i\in\mathbb{N}} \|\mathcal{T}_s\varphi_i(t)\|_{\ell^2_s}, \ \textit{for a.e. } t \in \mathbb{T}^n. \tag{2.2}$$

Proof. Let Y be an arbitrary shift-invariant space with the range function J_s and associated projections P_{J_s}. We will consider the function

$$F \in \mathcal{T}_s Y$$

defined as follows. If $Y = \{\mathbf{0}\}$, then $\mathcal{T}_s Y = \{\mathbf{0}\}$. Otherwise, we define $a_k \in H(\mathbb{T}^n, \ell^2_s)$ by

$$a_k(t) := \begin{cases} \frac{P_{J_s}(t)\mathrm{e}_{\pi(k)}}{\|P_{J_s}(t)\mathrm{e}_{\pi(k)}\|_{\ell^2_s}}, & t \in A_k, \\ 0, & \text{otherwise}, \end{cases}$$

where $\pi : \mathbb{N} \to \mathbb{Z}^n$ is bijection and $A_k := \{t \in \mathbb{T}^n : P_{J_s}(t)\mathrm{e}_{\pi(k)} \neq \mathbf{0}\}$, $k \in \mathbb{N}$. Let $(B_k)_{k\in\mathbb{N}}$ be a sequence of sets such that

$$B_1 := A_1, \qquad B_{k+1} := A_{k+1} \backslash \bigcup_{j=1}^{k} A_j,\ k \in \mathbb{N},$$

and define

$$F := \sum_{k\in\mathbb{N}} a_k \mathbf{1}_{B_k}.$$

Then, since $\sigma_Y = \bigcup_{k\in\mathbb{N}} A_k$, we have $F(t) \in J_s(t)$ and $\|F(t)\|_{\ell^2_s} = \mathbf{1}_{\sigma_Y}(t)$ for a.e. $t \in \mathbb{T}^n$. Thus, it is clear that $\varphi := \mathcal{T}_s^{-1} F$ is a quasi-orthogonal generator of $S_s(\varphi) \subset Y$, and $\sigma_{S_s(\varphi)} = \sigma_Y$.

It is easy seen that

$$\mathcal{T}_s(Y \ominus S_s(\varphi)) = \left\{Q \in H(\mathbb{T}^n, \ell^2_s) : Q(t) \in J_s(t), \langle F(t), Q(t)\rangle_{\ell^2_s} = 0 \text{ a.e. } t \in \mathbb{T}^n\right\}.$$

If $Q \in \mathcal{T}_s(Y \ominus S_s(\varphi))$, then

$$\langle Q(t), \mathrm{e}_{\pi(k)}\rangle_{\ell^2_s} = \langle Q(t), P_{J_s}(t)\mathrm{e}_{\pi(k)}\rangle_{\ell^2_s} = 0, \ \text{ for a.e. } t \in \mathbb{T}^n,\ k = \overline{1, k_0}, \quad (2.3)$$

where $k_0 = \min\{k \in \mathbb{N} : |A_k| \neq 0\}$.

Next, by induction on k, we will construct a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of quasi-orthogonal generators. Let $\varphi_1 := \mathcal{T}_s^{-1} F(V_s)$, and suppose that we have already constructed functions $\varphi_1, \dots, \varphi_k$, for some $k \in \mathbb{N}$, with properties:

(1) the function $\varphi_i \in V_s$ is a quasi-orthogonal generator of $V_s^i = S_s(\varphi_i)$, for $i = 1, \dots, k$;
(2) $V_s^i \perp V_s^j$ for $i \neq j$;
(3) for $Q \in \mathcal{T}_s W_s^k$, where

$$W_s^k = V_s \ominus \left(\bigoplus_{i=1}^{k} V_s^i\right), \quad (2.4)$$

we have $\langle Q(t), \mathrm{e}_{\pi(i)}\rangle_{\ell^2_s} = 0$ for all $i = 1, \dots, k$ and a.e. $t \in \mathbb{T}^n$.

Set $\varphi_{k+1} := \mathcal{T}_s^{-1} F(W_s^k)$. Now, the set $\{\varphi_1, \dots, \varphi_{k+1}\}$ satisfies the properties (1) – (3). Namely, (1) follows from the fact that $\|\mathcal{T}_s \varphi_{k+1}(t)\|_{\ell^2_s} = \mathbf{1}_{\sigma_{W_s^k}}(t)$; (2) is a consequence of the fact that $V_s^{k+1} \subset W_s^k$ and the equality (2.4) and, finally, (2.3) implies (3).

Take $Q \in \mathcal{T}_s\big(\bigcap_{i=1}^{+\infty} W_s^i\big)$. Then, from (3), we deduce that $\langle Q(t), \mathrm{e}_{\pi(i)}\rangle_{\ell^2_s} = 0$ for a.e. $t \in \mathbb{T}^n$ and all $i \in \mathbb{N}$. Hence, $\bigcap_{i=1}^{+\infty} W_s^i = \{\mathbf{0}\}$. Thus (2.1) follows. Next, since $W_s^{i+1} \subset W_s^i$, we conclude that $\sigma_{V_s^{i+1}} = \sigma_{W_s^{i+1}} \subset \sigma_{W_s^i} = \sigma_{V_s^i}$. Finally, (2.2) follows from (2.1). □

Unless otherwise stated, in the remainder of the paper, we assume $\oplus$ to be an orthogonal sum.

Proposition 2.1 ([5], Lemma 2.7). *Let J_{V_s} and J_{U_s} be the range functions for the shift-invariant subspaces V_s, U_s of H^s, respectively.*

(1) *The orthogonal complement of V_s, denoted by $V_s^{\perp}$, is a shift-invariant space and $J_{V_s^{\perp}}(t) = (J_{V_s}(t))^{\perp}$ for a.e. $t \in \mathbb{T}^n$.*
(2) *If $J_{V_s}(t) = J_{U_s}(t)$ for a.e. $t \in \mathbb{T}^n$, then $V_s = U_s$.*
(3) *The space $V_s \cap U_s$ is a shift-invariant space and $J_{V_s\cap U_s}(t) = J_{V_s}(t) \cap J_{U_s}(t)$, for a.e. $t \in \mathbb{T}^n$.*

Proof. (1) Let $f \in V_s^{\perp}$. Since $f \notin V_s$, we have $T_k f \notin V_s$ for every $k \in \mathbb{Z}^n$. Consequently, $T_k f \in V_s^{\perp}$, for every $k \in \mathbb{Z}^n$, i.e. $V_s^{\perp}$ is a shift-invariant space. Furthermore, for every $F \in H(\mathbb{T}^n, \ell_s^2)$, we have

$$F(t) \notin J_{V_s}(t) \text{ if and only if } F(t) \in (J_{V_s}(t))^{\perp}, \text{ for a.e. } t \in \mathbb{T}^n.$$

By Theorem 2.1, we get

$$\begin{aligned} F(t) \notin J_{V_s}(t) &\text{ if and only if } \mathcal{T}_s^{-1}F \notin V_s \\ &\text{ if and only if } F(t) \in J_{V_s^{\perp}}(t), \text{ for a.e. } t \in \mathbb{T}^n. \end{aligned}$$

Hence, $J_{V_s^{\perp}}(t) = (J_{V_s}(t))^{\perp}$ for a.e. $t \in \mathbb{T}^n$.

(2) If $J_{V_s}(t) = J_{U_s}(t)$ for a.e. $t \in \mathbb{T}^n$, then $M_{J_{V_s}} = M_{J_{U_s}}$. Since $\mathcal{T}_s V_s = M_{J_{V_s}}$ and $\mathcal{T}_s U_s = M_{J_{U_s}}$, we get $V_s = U_s$.

(3) Obviously, $V_s \cap U_s$ is a shift-invariant space. By Theorem 2.1, for $F \in H(\mathbb{T}^n, \ell_s^2)$, we have

$$\begin{aligned} F(t) \in J_{V_s\cap U_s}(t) &\text{ for a.e. } t \in \mathbb{T}^n \\ &\text{if and only if } \mathcal{T}_s^{-1}F \in V_s \cap U_s \\ &\text{if and only if } \mathcal{T}_s^{-1}F \in V_s \text{ and } \mathcal{T}_s^{-1}F \in U_s \\ &\text{if and only if } F(t) \in J_{V_s}(t) \text{ and } F(t) \in J_{U_s}(t) \text{ for a.e. } t \in \mathbb{T}^n \\ &\text{if and only if } F(t) \in J_{V_s}(t) \cap J_{U_s}(t) \text{ for a.e. } t \in \mathbb{T}^n. \end{aligned}$$

Thus, $J_{V_s\cap U_s}(t) = J_{V_s}(t) \cap J_{U_s}(t)$, for a.e. $t \in \mathbb{T}^n$. □

Proposition 2.2 ([5], Lemma 2.9). *Let $V_s \subset H^s$ be a shift-invariant space with corresponding range function J_s such that* $\dim J_s(t) < +\infty$ *for a.e. $t \in \mathbb{T}^n$. Then, there exist functions $\varphi_i \in H^s$, $i \in \mathbb{N}$, and measurable sets $(A_d)_{d\in\mathbb{N}_0}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, $\bigcup_{d\in\mathbb{N}_0} A_d = \mathbb{T}^n$ satisfying:*

(1) *$\{T_k\varphi_i : i \in \mathbb{N}, k \in \mathbb{Z}^n\}$ is a Parseval frame for V_s;*
(2) *for $i > d$, $\mathcal{T}_s\varphi_i(t) = \mathbf{0}$, for a.e. $t \in A_d$;*
(3) *$\{\mathcal{T}_s\varphi_1(t), \dots, \mathcal{T}_s\varphi_d(t)\}$ is an orthonormal basis for $J_s(t)$, for a.e. $t \in A_d$;*
(4) $\dim J_s(t) = d$ *for a.e. $t \in A_d$.*

Proof. By Theorem 2.3, there exist $\varphi_i \in H^s$, $i \in \mathbb{N}$, such that $\{T_k\varphi_i : i \in \mathbb{N}, k \in \mathbb{Z}^n\}$ is a Parseval frame for V_s. By Theorem 2.2, $\{\mathcal{T}_s\varphi_i(t) : i \in \mathbb{N}\}$ is a Parseval frame for $J_s(t)$ for a.e. $t \in \mathbb{T}^n$.

Since $\sigma_{V_s^{i+1}} \subset \sigma_{V_s^i}$ for every $i \in \mathbb{N}$, we can define a family of disjoint sets $(A_d)_{d\in\mathbb{N}_0}$ as follows: $A_0 := \mathbb{T}^n \backslash \sigma_{V_s}$ and $A_d := \sigma_{V_s^d} \backslash \sigma_{V_s^{d+1}}$ for $d > 0$. Since

$\dim J_s(t) < +\infty$, we have $\sum_{i\in\mathbb{N}} \|\mathcal{T}_s\varphi_i(t)\|_{\ell_s^2} < +\infty$, and thus $\bigcap_{i\in\mathbb{N}} \sigma_{V_s^i} = \emptyset$. Consequently, $\bigcup_{d\in\mathbb{N}_0} A_d = \mathbb{T}^n$.

It is obvious that (2) is true for $d > 0$. Further, by (1), Theorem 2.3 and Lemma 2.1, the assertion (3) follows, and thus $\dim J_s(t) = d$ for a.e. $t \in A_d$. For $d = 0$, we have $J_s(t) = \{\mathbf{0}\}$ for a.e. $t \in A_0$. □

3. Shift-preserving operators and range operators

Let V_s be a shift-invariant subspace of H^s with the range function J_s and associated projection P_{J_s} (see [7]). Following [17] we define the shift-preserving operator and the range operator.

We say that a bounded linear operator $L : V_s \to H^s$ is shift-preserving if $LT_k = T_kL$ for all $k \in \mathbb{Z}^n$.

A range operator on J_s (with values in ℓ_s^2) is a mapping $R_s : \mathbb{T}^n \to$ {set of bounded operators defined on closed subspaces of ℓ_s^2}, so that the domain of $R_s(t)$ equals $J_s(t)$ for a.e. $t \in \mathbb{T}^n$. The operator R_s is measurable if $t \mapsto R_s(t)P_{J_s}(t)$ is weakly operator measurable, i.e. $t \mapsto \langle R_s(t)P_{J_s}(t)a, b\rangle_{\ell_s^2}$ is measurable scalar function for each $a, b \in \ell_s^2$.

Now we transfer results from [17] related to L^2 spaces to H^s spaces.

Theorem 3.1 ([17], Proposition 4.2). *Let $\varphi \in H^s$ be a tight-frame generator of $S_s(\varphi)$ and an operator $L : S_s(\varphi) \to H^s$ be shift-preserving. Then, for any $g \in L_s^2(\mathbb{T}^n)$,*

$$\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)(g\Phi)(t) = g(t)\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)\Phi(t) \quad \textit{for a.e. } t \in \mathbb{T}^n, \tag{3.1}$$

where $\Phi = \mathcal{T}_s\varphi$.

Proof. For $\Phi = \mathcal{T}_s\varphi$, $\varphi \in S_s(\varphi)$, by Lemma 2.1, we have

$$\begin{aligned}\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)\big(\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\Phi\big) = \big(\mathcal{T}_s L T_k\big)\varphi &= \big(\mathcal{T}_s T_k L\big)\big(\mathcal{T}_s^{-1}\mathcal{T}_s\varphi\big) \\ &= \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)\Phi.\end{aligned}$$

Thus, for all polynomials $p_m(t) = \sum_{|k|\le m} a_k \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle}$, holds

$$\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)(p_m\Phi)(t) = p_m(t)\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)\Phi(t) \quad \text{for a.e. } t \in \mathbb{T}^n. \tag{3.2}$$

Using Lemma 2.1, the fact that φ is a tight-frame generator of $V_s = S_s(\varphi)$, and boundedness of L, we get

$$\begin{aligned}\int_{\mathbb{T}^n} |p_m(t)|^2 \big\|(\mathcal{T}_s L \mathcal{T}_s^{-1})\Phi(t)\big\|_{\ell_s^2}^2 \,\mathrm{d}t &= \int_{\mathbb{T}^n} \big\|(\mathcal{T}_s L \mathcal{T}_s^{-1})(p_m\Phi)(t)\big\|_{\ell_s^2}^2 \,\mathrm{d}t \\ &= \big\|(\mathcal{T}_s L \mathcal{T}_s^{-1})(p_m\Phi)\big\|_{H(\mathbb{T}^n,\ell_s^2)}^2 \\ &\le C^2 \big\|p_m\Phi\big\|_{H(\mathbb{T}^n,\ell_s^2)}^2 \\ &= C^2 \int_{\mathbb{T}^n} |p_m(t)|^2 \|\Phi(t)\|_{\ell_s^2}^2 \,\mathrm{d}t \\ &= C^2 \int_{\mathbb{T}^n} |p_m(t)|^2 \mathbf{1}_{\sigma_{S_s(\varphi)}}(t) \,\mathrm{d}t < +\infty. \end{aligned} \tag{3.3}$$

For every $r \in L^\infty(\mathbb{T}^n)$ there exists a sequence of polynomials $(p^i_{m_i})_{i\in\mathbb{N}}$ such that: $\|p^i_{m_i}\|_\infty \le \|r\|_\infty$, $i \in \mathbb{N}$, and $\lim_{i\to+\infty} p^i_{m_i}(t) = r(t)$, for a.e. $t \in \mathbb{T}^n$. Thus, from (3.3), we have

$$\int_{\mathbb{T}^n} |r(t)|^2 \big\|(\mathcal{T}_s L \mathcal{T}_s^{-1})\Phi(t)\big\|_{\ell_s^2}^2 \,\mathrm{d}t \le C^2 \int_{\mathbb{T}^n} |r(t)|^2 \|\Phi(t)\|_{\ell_s^2}^2 \,\mathrm{d}t,$$

so

$$\big\|(\mathcal{T}_s L \mathcal{T}_s^{-1})\Phi(t)\|_{\ell_s^2} \le C\|\Phi(t)\|_{\ell_s^2} \quad \text{for a.e. } t \in \mathbb{T}^n. \tag{3.4}$$

Therefore, we can choose a sequence $(p^i_{m_i})_{i\in\mathbb{N}}$ such that $p^i_{m_i} \to g$ in $L^2_s(\mathbb{T}^n)$ and additionally

$$\lim_{i\to+\infty} p^i_{m_i}(t) = g(t), \quad \lim_{i\to+\infty} \big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)(p^i_{m_i}\Phi)(t) = \big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)(g\Phi)(t), \tag{3.5}$$

for a.e. $t \in \mathbb{T}^n$. Finally, from (3.2), using (3.4) and (3.5), it follows that (3.1) holds for a general functions in $L^2_s(\mathbb{T}^n)$. □

Let us mention a consequence of the theorems 3.1 and 2.1.

Corollary 3.1. *Let $V_s \subset H^s$ be a shift-invariant space and let $L : V_s \to H^s$ be the shift-preserving operator. Then, for any $\Phi \in \mathcal{T}_s V_s$ and measurable function g so that $g\Phi \in H(\mathbb{T}^n, \ell^2_s)$ (and thus $g\Phi \in \mathcal{T}_s V_s$), holds*

$$\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)(g\Phi)(t) = g(t)\big(\mathcal{T}_s L \mathcal{T}_s^{-1}\big)\Phi(t), \quad \textit{for a.e. } t \in \mathbb{T}^n.$$

Theorem 3.2 ([17], Theorem 4.5). *Let $V_s \subset H^s$ be a shift-invariant space with the range function J_s.*

(1) *Let $L : V_s \to H^s$ be a shift-preserving operator. Then, there exists a measurable range operator R_s on J_s so that*

$$(\mathcal{T}_s L)f(t) = R_s(t)(\mathcal{T}_s f(t)) \quad \textit{for a.e. } t \in \mathbb{T}^n,\ f \in V_s. \tag{3.6}$$

(2) *Let R_s be a measurable range operator on J_s such that*

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_s(t)\| < +\infty.$$

Then, there exists a shift-preserving operator $L : V_s \to H^s$ that satisfies (3.6).

The correspondence between L and R_s is one-to-one under the convention that the range operators are identified if they are equal almost everywhere. Furthermore, $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|R_s(t)\| = \|L\|$.

Proof. (1) From Theorem 2.3, we have that $V_s = \oplus_{i=1}^{+\infty} S_s(\varphi_i)$, where φ_i is a tight-frame generator of $S_s(\varphi_i)$.
First, we observe $V_s^k := \oplus_{i=1}^{k} S_s(\varphi_i)$ with a range function J_s^k. Let $\Phi_i = \mathcal{T}_s\varphi_i$. Then $\{\Phi_1(t), ..., \Phi_k(t)\}\backslash\{\mathbf{0}\}$ (if $t \notin \sigma_{V_s^k}$, then $\Phi_i(t) = \mathbf{0}$, $i = 1, \dots, k$) is an orthonormal basis of $J_s^k(t)$ for a.e. $t \in \mathbb{T}^n$. We define $R_s^k(t) : J_s^k(t) \to \ell_s^2$ with

$$R_s^k(t)\left(\sum_{i=1}^{k} \alpha_i\Phi_i(t)\right) = \sum_{i=1}^{k} \alpha_i(\mathcal{T}_sL\mathcal{T}_s^{-1})\Phi_i(t), \quad \alpha_i \in \mathbb{C}, i = 1, \dots, k.$$

Note, by (3.4) that, it is well defined operator. Now, let $f \in V_s^k$ be arbitrary. Then there exist $f_i \in S_s(\varphi_i)$, $i = 1, ..., k$, such that $f = f_1 + \cdots + f_k$, and $\mathcal{T}_sf = \mathcal{T}_sf_1 + \cdots + \mathcal{T}_sf_k = g_1\Phi_1 + \cdots + g_k\Phi_k$, for some $g_i \in L_s^2(\mathbb{T}^n)$. Therefore, by Theorem 3.1, we have

$$\begin{aligned}(\mathcal{T}_sL)f(t) = (\mathcal{T}_sL\mathcal{T}_s^{-1})\left(\sum_{i=1}^{k} g_i\Phi_i\right)(t) &= \sum_{i=1}^{k} g_i(t)(\mathcal{T}_sL\mathcal{T}_s^{-1})\Phi_i(t) \\ &= \sum_{i=1}^{k} g_i(t)R_s^k(t)\big(\Phi_i(t)\big) \\ &= \sum_{i=1}^{k} R_s^k(t)\big(g_i(t)\Phi_i(t)\big) \\ &= R_s^k(t)\big(\mathcal{T}_sf(t)\big). \end{aligned} \tag{3.7}$$

Since $\mathcal{T}_s$ is the isometry, we conclude that R_s^k is measurable.

Let $\|L\| \le C$, where C is some positive constant. We will proof that $\|R_s^k(t)\| \le C$ for a.e. $t \in \mathbb{T}^n$. First, we will proof that

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_s^k(t)(\Psi_\alpha(t))\|_{\ell_s^2} \le C, \tag{3.8}$$

where $\Psi_\alpha \in H(\mathbb{T}^n, \ell_s^2)$ is given by $\Psi_\alpha(t) := \sum_{i=1}^{k} \alpha_i\Phi_i(t)$ for $\alpha = (\alpha_1, ..., \alpha_k) \in S^k := \{\alpha \in \mathbb{C}^k : |\alpha_1|^2 + \cdots + |\alpha_k|^2 = 1\}$. Obviously, $\|\Psi_\alpha(t)\|_{\ell_s^2} = 1$.

To obtain a contradiction, suppose that (3.8) were false. Then, there exist $\varepsilon > 0$ and an nonempty measurable set $A \subset \mathbb{T}^n$ such that for $t \in A$, $\|R_s^k(t)(\Psi_\alpha(t))\|_{\ell_s^2} > C + \varepsilon$. Let $\Psi = \Psi_\alpha\mathbf{1}_A$ and $\psi := \mathcal{T}_s^{-1}\Psi \in V_s^k$. Since $\mathcal{T}_s$ is isometry and $\|L\| \le C$, we get

$$\|(\mathcal{T}_sL)\psi\|_{H(\mathbb{T}^n,\ell_s^2)} = \|L\psi\|_{H^s} \le C\|\psi\|_{H^s} = C\|\Psi\|_{H(\mathbb{T}^n,\ell_s^2)}.$$

But, by (3.7),

$$\begin{aligned}\|(\mathcal{T}_s L)\psi\|^2_{H(\mathbb{T}^n,\ell^2_s)} &= \int_{\mathbb{T}^n} \|R^k_s(t)(\Psi(t))\|^2_{\ell^2_s}\,\mathrm{d}t = \int_A \|R^k_s(t)(\Psi_\alpha(t))\|^2_{\ell^2_s}\,\mathrm{d}t\\ &\geq (C+\varepsilon)^2 \int_A \mathrm{d}t = (C+\varepsilon)^2 \int_A \|\Psi_\alpha(t)\|^2_{\ell^2_s}\,\mathrm{d}t\\ &= (C+\varepsilon)^2 \|\Psi\|^2_{H(\mathbb{T}^n,\ell^2_s)},\end{aligned}$$

which is a contradiction. Thus, (3.8) is true. Now, by (3.8), for a dense subset $(\alpha_n)_{n\in\mathbb{N}}$ of S^k, we have

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R^k_s(t)\| &= \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \sup_{\alpha\in S^k} \|R^k_s(t)(\Psi_\alpha(t))\|_{\ell^2_s}\\ &= \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \sup_{n\in\mathbb{N}} \|R^k_s(t)(\Psi_{\alpha_n}(t))\|_{\ell^2_s} \leq C.\end{aligned}$$

Hence, $\|R^k_s(t)\| \leq C$ for a.e. $t\in\mathbb{T}^n$.

Let $\ell \leq k$. Since $R^\ell_s(t) = R^k_s(t)|_{J^\ell_s(t)}$, we define $R_s(t) : \bigcup_{\ell\in\mathbb{N}} J^\ell_s(t) \to \ell^2_s$ with $R_s(t)(a) = R^\ell_s(t)(a)$, $a\in J^\ell_s(t)$ for some $\ell\in\mathbb{N}$. Then,

$$\|R_s(t)(a)\|_{\ell^2_s} \leq C\|a\|_{\ell^2_s}, \quad a \in \bigcup_{\ell\in\mathbb{N}} J^\ell_s(t).$$

Therefore, we can uniquely extend $R_s(t)$ to $R_s(t) : J_s(t) \to \ell^2_s$ with $\|R_s(t)\| \leq C$, using the fact that

$$\overline{\bigcup_{\ell\in\mathbb{N}} J^\ell_s(t)} = J_s(t).$$

Finally, let us show that (3.6) holds. Choose any $f\in V_s$ and $(f_k)_{k\in\mathbb{N}}$, $f_k\in V^k_s$, such that $\lim_{k\to+\infty} f_k = f$ in H^s, $\lim_{k\to+\infty} \mathcal{T}_s f_k(t) = \lim_{k\to+\infty} \mathcal{T}_s f(t)$ and $\lim_{k\to+\infty}(\mathcal{T}_s L) f_k(t) = \lim_{k\to+\infty}(\mathcal{T}_s L) f(t)$ for a.e. $t\in\mathbb{T}^n$. Thus, for this sequence, by the previous construction and (3.7), we have

$$(\mathcal{T}_s L) f_k(t) = R_s(t)(\mathcal{T}_s f_k(t)).$$

Let $k\to+\infty$. It follows that (3.6) holds.

(2) Let R_s be a measurable range operator on J_s such that

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_s(t)\| = C < +\infty.$$

Then, $R_s(t)(\mathcal{T}_s f(t))$ is also measurable operator and

$$\begin{aligned}\|R_s(\mathcal{T}_s f)\|^2_{H(\mathbb{T}^n,\ell^2_s)} &= \int_{\mathbb{T}^n} \|R_s(t)(\mathcal{T}_s f(t))\|^2_{\ell^2_s}\,\mathrm{d}t\\ &\leq \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_s(t)\|^2 \int_{\mathbb{T}^n} \|\mathcal{T}_s f(t)\|^2_{\ell^2_s}\,\mathrm{d}t\\ &= C^2\|\mathcal{T}_s f\|^2_{H(\mathbb{T}^n,\ell^2_s)} = C^2\|f\|^2_{H^s}. \qquad (3.9)\end{aligned}$$

For $L : V_s \to H^s$ given by $Lf = \mathcal{T}_s^{-1} R_s(\mathcal{T}_s f)$, we have: L is linear; $\|Lf\|_{H^s} \le C\|f\|_{H^s}$ (by (3.9)); L satisfies (3.6) and since

$$\begin{aligned}(\mathcal{T}_s L T_k) f(t) = R_s(t)(\mathcal{T}_s T_k f(t)) &= R_s(t)(\mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle}\mathcal{T}_s f(t)) \\ &= \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} R_s(t)(\mathcal{T}_s f(t)) \\ &= \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle}\mathcal{T}_s(Lf)(t) \\ &= (\mathcal{T}_s T_k L) f(t),\end{aligned}$$

it follows that L is a shift-preserving operator.

Finally, from (3.6) it follows immediately that there is a unique correspondence between the operators L and R_s. □

Theorem 3.3 ([17], Theorem 4.6). *Let $L : V_s \to H^s$ be a shift-preserving operator with the corresponding range operator R_s on J_s. Then,*

$$\|Lf\|_{H^s} \ge C\|f\|_{H^s}, \quad f \in V_s, \tag{3.10}$$

if and only if

$$\|R_s(t)(a)\|_{\ell^2_s} \ge C\|a\|_{\ell^2_s}, \quad a \in J_s(t), \text{ for a.e. } t \in \mathbb{T}^n, \tag{3.11}$$

where $C > 0$ is a constant.

Proof. If (3.11) holds, then using (3.6) we have

$$\begin{aligned}\|Lf\|^2_{H^s} &= \|(\mathcal{T}_s L) f\|^2_{H(\mathbb{T}^n,\ell^2_s)} = \int_{\mathbb{T}^n} \|R_s(t)(\mathcal{T}_s f(t))\|^2_{\ell^2_s}\,\mathrm{d}t \\ &\ge \int_{\mathbb{T}^n} C^2\|\mathcal{T}_s f(t)\|^2_{\ell^2_s}\,\mathrm{d}t = C^2\|\mathcal{T}_s f\|^2_{H(\mathbb{T}^n,\ell^2_s)} = C^2\|f\|^2_{H^s}, \quad f \in V_s.\end{aligned}$$

Assume, now, that (3.10) holds. Let $\{a_1, a_2, \ldots\}$ be a dense subset of ℓ^2_s. We will prove that for a.e. $t \in \mathbb{T}^n$,

$$\|R_s(t)(P_{J_s}(t)a_i)\|_{\ell^2_s} \ge C\|P_{J_s}(t)a_i\|_{\ell^2_s}, \quad i \in \mathbb{N}. \tag{3.12}$$

Suppose that (3.12) is not valid. Then there exist a nonempty measurable set $A \subset \mathbb{T}^n$, $i_0 \in \mathbb{N}$, and $\varepsilon > 0$ such that, for $t \in A$ we have

$$\|R_s(t)(P_{J_s}(t)a_{i_0})\|_{\ell^2_s} \le (C-\varepsilon)\|P_{J_s}(t)a_{i_0}\|_{\ell^2_s}.$$

Let $f \in V_s$ be a function such that $\mathcal{T}_s f(t) = \mathbf{1}_A(t)P_{J_s}(t)a_{i_0}$. Then, using (3.6) again, we have

$$\begin{aligned}\|Lf\|^2_{H^s} = \|(\mathcal{T}_s L) f\|^2_{H(\mathbb{T}^n,\ell^2_s)} &= \int_{\mathbb{T}^n} \|R_s(t)(\mathcal{T}_s f(t))\|^2_{\ell^2_s}\,\mathrm{d}t \\ &\le (C-\varepsilon)^2 \int_A \|P_{J_s}(t)a_{i_0}\|^2_{\ell^2_s}\,\mathrm{d}t \\ &= (C-\varepsilon)^2 \int_{\mathbb{T}^n} \|\mathcal{T}_s f(t)\|^2_{\ell^2_s}\,\mathrm{d}t \\ &= (C-\varepsilon)^2\|f\|^2_{H^s},\end{aligned}$$

contrary to (3.10). □

Corollary 3.2. *Let L be a shift-preserving operator and let R_s be its corresponding range operator. Then, L is an isometry if and only if $R_s(t)$ is an isometry, for a.e. $t \in \mathbb{T}^n$.*

Proof. Using (3.6) we get

$$\|(\mathcal{T}_s L)f\|^2_{H(\mathbb{T}^n,\ell^2_s)} = \int_{\mathbb{T}^n} \|(\mathcal{T}_s L)f(t)\|^2_{\ell^2_s}\,\mathrm{d}t = \int_{\mathbb{T}^n} \|R_s(t)(\mathcal{T}_s f(t))\|^2_{\ell^2_s}\,\mathrm{d}t$$
$$= \|R_s(\mathcal{T}_s f)\|^2_{H(\mathbb{T}^n,\ell^2_s)}.$$

Thus, since

$$\|R_s(\mathcal{T}_s f)\|_{H(\mathbb{T}^n,\ell^2_s)} = \|(\mathcal{T}_s L)f\|_{H(\mathbb{T}^n,\ell^2_s)} = \|Lf\|_{H^s} \le C\|f\|_{H^s} = C\|\mathcal{T}_s f\|_{H(\mathbb{T}^n,\ell^2_s)},$$

$C > 0$, the assertion follows. □

Theorem 3.4 ([17], Theorem 4.8). *Let $V_s \subset H^s$ be a shift-invariant space with the associated range function J_s and $L : V_s \to V_s$ be a shift-preserving operator with corresponding range operator R_s.*

(1) *The adjoint operator $L^* : V_s \to V_s$ is also shift-preserving and with $R^*_s(t) = (R_s(t))^*$, for a.e. $t \in \mathbb{T}^n$, is given the corresponding range operator R^*_s.*
(2) *The operator L is self-adjoint and $\sigma(L) \subseteq [A, B]$ if and only if $R_s(t)$ is self-adjoint and $\sigma(R_s(t)) \subseteq [A, B]$ for a.e. $t \in \mathbb{T}^n$, where A, $B \in \mathbb{R}$ such that $A \le B$.*
(3) *The operator L is unitary if and only if $R_s(t)$ is unitary operator for a.e. $t \in \mathbb{T}^n$.*

Proof. (1) It is obvious that R^*_s is measurable and uniformly bounded on J_s. From Theorem 3.2 we have that there exists a shift-preserving operator $L^\bullet : V_s \to H^s$ such that $(\mathcal{T}_s L^\bullet)f(t) = R^*_s(t)(\mathcal{T}_s f(t))$, $f \in V_s$. Thus, for f, $g \in V_s$, we get

$$\langle Lf, g\rangle_{H^s} = \langle(\mathcal{T}_s L)f, \mathcal{T}_s g\rangle_{H(\mathbb{T}^n,\ell^2_s)} = \int_{\mathbb{T}^n} \langle R_s(t)(\mathcal{T}_s f(t)), \mathcal{T}_s g(t)\rangle_{\ell^2_s}\,\mathrm{d}t$$
$$= \int_{\mathbb{T}^n} \langle \mathcal{T}_s f(t), R^*_s(t)(\mathcal{T}_s g(t))\rangle_{\ell^2_s}\,\mathrm{d}t$$
$$= \langle \mathcal{T}_s f, (\mathcal{T}_s L^\bullet)g\rangle_{H(\mathbb{T}^n,\ell^2_s)} = \langle f, L^\bullet g\rangle_{H^s}.$$

(2) From the part (1), we have $L^* = L$ if and only if $R^*_s(t) = R_s(t)$, for a.e. $t \in \mathbb{T}^n$. Let $\sigma(L) \subseteq [A, B]$, i.e. for all $f \in V_s$ we have

$$A\|f\|^2_{H^s} \le \langle Lf, f\rangle_{H^s} = \langle \mathcal{T}_s Lf, \mathcal{T}_s f\rangle_{H(\mathbb{T}^n,\ell^2_s)}$$
$$= \int_{\mathbb{T}^n} \langle R_s(t)(\mathcal{T}_s f(t)), \mathcal{T}_s f(t)\rangle_{\ell^2_s}\,\mathrm{d}t \le B\|f\|^2_{H^s}. \quad (3.13)$$

Analysis similar to that in the proof of Theorem 3.3 shows that the assertion holds.

In the opposite direction, suppose that $\sigma(R_s(t)) \subseteq [A, B]$ for a.e. $t \in \mathbb{T}^n$, i.e. for $f \in V_s$

$$A\|\mathcal{T}_s f(t)\|^2_{\ell^2_s} \le \big\langle R_s(t)(\mathcal{T}_s f(t)), \mathcal{T}_s f(t)\big\rangle_{\ell^2_s} \le B\|\mathcal{T}_s f(t)\|^2_{\ell^2_s}, \quad \text{for a.e. } t \in \mathbb{T}^n.$$

We obtain (3.13) by the integration of above inequalities over $\mathbb{T}^n$.

(3) The statement follows from (2). L is unitary, $\sigma(LL^*) = \sigma(L^*L) = \{1\}$, if and only if $\sigma(R_s(t)R_s^*(t)) = \sigma(R_s^*(t)R_s(t)) = \{1\}$ for a.e. $t \in \mathbb{T}^n$, i.e. $R_s(t)$ is unitary for a.e. $t \in \mathbb{T}^n$. $\square$

Similarly as in [17], we have the next propositions.

Proposition 3.1. *Let $V_s \subset H^s$ be a shift-invariant space and let L be its shift-preserving operator. For $\widetilde{V_s} = \overline{L(V_s)}$, we have $\dim_{\widetilde{V_s}}(t) \leq \dim_{V_s}(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. Let $V_s = S_s(\mathcal{A}_{I,s})$. Using the theorems 3.2 and 2.1, the range function $\widetilde{J_s}$ of $\widetilde{V_s} = S_s\big(\{L\varphi : \varphi \in \mathcal{A}_{I,s}\}\big)$ satisfies

$$\begin{aligned}\widetilde{J_s}(t) &= \overline{\operatorname{span}}\big\{\mathcal{T}_s\phi(t) : \phi \in \{L\varphi : \varphi \in \mathcal{A}_{I,s}\}\big\} = \overline{\operatorname{span}}\big\{(\mathcal{T}_s L)\varphi(t) : \varphi \in \mathcal{A}_{I,s}\big\} \\ &= \overline{\operatorname{span}}\big\{R_s(t)(\mathcal{T}_s\varphi(t)) : \varphi \in \mathcal{A}_{I,s}\big\} = \overline{R_s(t)(J_s(t))}, \text{ for a.e. } t \in \mathbb{T}^n.\end{aligned}$$

Thus, we have that $\dim \widetilde{J_s}(t) \leq \dim J_s(t)$.

On the other hand, by Theorem 2.3, $\dim_{V_s}(t) = \sum_{i \in \mathbb{N}} \|\mathcal{T}_s\varphi_i(t)\|_{\ell^2_s}$, for a.e. $t \in \mathbb{T}^n$, which completes the proof. $\square$

Proposition 3.2. *Let $V_s, \widetilde{V_s} \subset H^s$ be the shift-invariant spaces. Then,*

$$\dim_{V_s}(t) = \dim_{\widetilde{V_s}}(t) \quad \textit{for a.e. } t \in \mathbb{T}^n$$

if and only if there exists an operator $L : V_s \to \widetilde{V_s}$ which is a shift-preserving isomorphism (or isometry).

Proof. Suppose that V_s and $\widetilde{V_s}$ are shift-invariant spaces such that $\dim_{V_s}(t) = \dim_{\widetilde{V_s}}(t)$ for a.e. $t \in \mathbb{T}^n$. Using Theorem 2.3, the spaces V_s and $\widetilde{V_s}$ can be decomposed as

$$V_s = \bigoplus_{i \in \mathbb{N}} S_s(\varphi_i), \quad \widetilde{V_s} = \bigoplus_{i \in \mathbb{N}} S_s(\widetilde{\varphi}_i),$$

where φ_i and $\widetilde{\varphi}_i$ are tight-frame generators, and $\sigma_{S_s(\varphi_i)} = \sigma_{S_s(\widetilde{\varphi}_i)}$, $i \in \mathbb{N}$. Let $L_i : S_s(\varphi_i) \to S_s(\widetilde{\varphi}_i)$ be defined by $L_i(T_k\varphi_i) = T_k\widetilde{\varphi}_i$, $i \in \mathbb{N}$. Now, for arbitrary $(c_k)_{k \in \mathbb{Z}^n} \in \ell^2_s$ with finite number of elements which are different from zero, we have

$$\Big\| \sum_{k \in \mathbb{Z}^n} c_k T_k \varphi_i \Big\|^2_{H^s} = \Big\| \sum_{k \in \mathbb{Z}^n} c_k \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} \mathcal{T}_s\varphi_i(t) \Big\|^2_{H(\mathbb{T}^n, \ell^2_s)}$$

$$= \int_{\mathbb{T}^n} \Big| \sum_{k\in\mathbb{Z}^n} c_k \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} \Big|^2 \|\mathcal{T}_s\varphi_i(t)\|^2_{\ell^2_s}\,\mathrm{d}t$$
$$= \int_{\mathbb{T}^n} \Big| \sum_{k\in\mathbb{Z}^n} c_k \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} \Big|^2 \|\mathcal{T}_s\widetilde{\varphi}_i(t)\|^2_{\ell^2_s}\,\mathrm{d}t$$
$$= \Big\| \sum_{k\in\mathbb{Z}^n} c_k T_k \widetilde{\varphi}_i \Big\|^2_{H^s} = \Big\| L_i\Big(\sum_{k\in\mathbb{Z}^n} c_k T_k\varphi_i\Big)\Big\|^2_{H^s}.$$

Therefore, L_i is a shift-preserving isometry. Then, the operator $L := \bigoplus_{i\in\mathbb{N}} L_i$ has the desired properties.

In the opposite direction, assume that $L : V_s \to \widetilde{V_s}$ is a shift-preserving isomorphism. Then, applying Proposition 3.1 to $L : V_s \to \widetilde{V_s}$ and $L^{-1} : \widetilde{V_s} \to V_s$, we get $\dim_{V_s}(t) = \dim_{\widetilde{V_s}}(t)$, for a.e. $t \in \mathbb{T}^n$. □

4. The frame operator

Let $E_s(\mathcal{A}_{I,s})$ be a Bessel family for $S_s(\mathcal{A}_{I,s})$. The operator

$$K : S_s(\mathcal{A}_{I,s}) \to \ell^2_s(\mathbb{Z}^n \times I)$$

given by $Kf = \big(\frac{\langle f, T_k\varphi_i\rangle_{H^s}}{\mu_s(k+i)}\big)_{(k,i)\in\mathbb{Z}^n\times I}$, has the dual $K^* : \ell^2_s(\mathbb{Z}^n \times I) \to S_s(\mathcal{A}_{I,s})$ defined by

$$K^*c = \sum_{(k,i)\in\mathbb{Z}^n\times I} c_{k,i} T_k\varphi_i \mu_s(k+i).$$

In order to distinguish the shift-invariant operator defined by K^*K and a general shift-invariant operator L defined in the previous section, we use notation $L_0 = K^*K$. The frame operator $L_0 : S_s(\mathcal{A}_{I,s}) \to S_s(\mathcal{A}_{I,s})$ given by $L_0 = K^*K$ is self-adjoint and

$$L_0 f = \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle f, T_k\varphi_i\rangle_{H^s} T_k\varphi_i,$$

where the convergence is unconditional in H^s.

Let $t \in \mathbb{T}^n$ be fixed and consider $(\mathcal{T}_s\varphi_i(t))_{i\in I}$. The corresponding Gramian $G_s : \ell^2(I) \to \ell^2(I)$ and the dual Gramian $\widetilde{G}_s : \ell^2_s(\mathbb{Z}^n) \to \ell^2_s(\mathbb{Z}^n)$ is defined by $G_s = N_s^* N_s$ and $\widetilde{G}_s = N_s N_s^*$, respectively, where $N_s : \ell^2(I) \to \ell^2_s(\mathbb{Z}^n)$ is given by $N_s(c) = \sum_{i\in I} c_i \mathcal{T}_s\varphi_i(t)$, and $N_s^* : \ell^2_s(\mathbb{Z}^n) \to \ell^2(I)$ is adjoint of N_s given by $N_s^*(a) = \big(\langle a, \mathcal{T}_s\varphi_i(t)\rangle_{\ell^2_s}\big)_{i\in I}$.

Remark 4.1. *Obviously, if $(\mathcal{T}_s\varphi_i)_{i\in I}$ is a Bessel family, then the operator N_s (i.e. N_s^*) is bounded, and vice versa.*

The following two assertions are necessary for the proof of Theorem 4.2.

Lemma 4.1 ([7]). *Let $E_s(\mathcal{A}_{I,s})$ be a Bessel family. Then, for all $f \in \mathcal{A}_{I,s}$, we have*

$$\sum_{\varphi\in\mathcal{A}_{I,s}} \sum_{k\in\mathbb{Z}^n} \left|\langle T_k\varphi, f\rangle_{H^s}\right|^2 = \sum_{\varphi\in\mathcal{A}_{I,s}} \int_{\mathbb{T}^n} \left|\left\langle \mathcal{T}_s\varphi(t), \mathcal{T}_s f(t)\right\rangle_{\ell^2_s}\right|^2 \mathrm{d}t.$$

Theorem 4.1 ([17], **Theorem 5.1**). *Let $E_s(\mathcal{A}_{I,s})$ be a Bessel family for $S_s(\mathcal{A}_{I,s})$ and let J_s be the range function of $S_s(\mathcal{A}_{I,s})$. Then, $L_0 = K^*K$ is self-adjoint, shift-preserving operator with the range operator $R_s(t) = \widetilde{G}_s(t)|_{J_s(t)}$, where $\widetilde{G}_s(t)$ is the dual Gramian of $\{\mathcal{T}_s\varphi_i(t) : i \in I\}$ for a.e. $t \in \mathbb{T}^n$.*

Proof. Since

$$\begin{aligned} L_0T_\ell f &= \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle T_\ell f, T_k\varphi_i\rangle_{H^s} T_k\varphi_i = \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle f, T_{k-\ell}\varphi_i\rangle_{H^s} T_k\varphi_i \\ &= \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle f, T_k\varphi_i\rangle_{H^s} T_{k+\ell}\varphi_i, \end{aligned}$$

for all $\ell \in \mathbb{Z}^n$, we have $L_0T_\ell = T_\ell L_0$, i.e. L_0 is shift-preserving. Further,

$$\begin{aligned} \|Kf\|^2_{\ell^2_s} = \langle Kf, Kf\rangle_{\ell^2_s} &= \langle L_0 f, f\rangle_{H^s} \\ &= \langle (\mathcal{T}_s L_0)f, \mathcal{T}_s f\rangle_{H(\mathbb{T}^n,\ell^2_s)} \\ &= \int_{\mathbb{T}^n} \left\langle \widetilde{R}_s(t)\big(\mathcal{T}_s f(t)\big), \mathcal{T}_s f(t)\right\rangle_{\ell^2_s} \mathrm{d}t, \end{aligned} \tag{4.1}$$

for $f \in S_s(\mathcal{A}_{I,s})$, where $\widetilde{R}_s$ is the range operator of L_0. By Lemma 4.1, we get

$$\begin{aligned} \|Kf\|^2_{\ell^2_s} &= \left\| \left(\frac{\langle f, T_k\varphi_i\rangle_{H^s}}{\mu_s(k+i)} \right)_{(k,i)\in\mathbb{Z}^n\times I} \right\|^2_{\ell^2_s} \\ &= \sum_{(k,i)\in\mathbb{Z}^n\times I} \left|\langle f, T_k\varphi_i\rangle_{H^s}\right|^2 \\ &= \sum_{i\in I} \int_{\mathbb{T}^n} \left|\left\langle \mathcal{T}_s f(t), \mathcal{T}_s\varphi_i(t)\right\rangle_{\ell^2_s}\right|^2 \mathrm{d}t \\ &= \int_{\mathbb{T}^n} \sum_{i\in I} \left\langle \mathcal{T}_s f(t), \mathcal{T}_s\varphi_i(t)\right\rangle_{\ell^2_s} \overline{\left\langle \mathcal{T}_s f(t), \mathcal{T}_s\varphi_i(t)\right\rangle_{\ell^2_s}} \mathrm{d}t \\ &= \int_{\mathbb{T}^n} \left\langle \left(\left\langle \mathcal{T}_s f(t), \mathcal{T}_s\varphi_i(t)\right\rangle_{\ell^2_s}\right)_{i\in I}, \left(\left\langle \mathcal{T}_s f(t), \mathcal{T}_s\varphi_i(t)\right\rangle_{\ell^2_s}\right)_{i\in I} \right\rangle_{\ell^2} \mathrm{d}t \\ &= \int_{\mathbb{T}^n} \left\langle N_s^*(\mathcal{T}_s f(t)), N_s^*(\mathcal{T}_s f(t))\right\rangle_{\ell^2} \mathrm{d}t \\ &= \int_{\mathbb{T}^n} \left\langle N_s N_s^*(\mathcal{T}_s f(t)), \mathcal{T}_s f(t)\right\rangle_{\ell^2_s} \mathrm{d}t \\ &= \int_{\mathbb{T}^n} \left\langle \widetilde{G}_s(t)\mathcal{T}_s f(t), \mathcal{T}_s f(t)\right\rangle_{\ell^2_s} \mathrm{d}t. \end{aligned} \tag{4.2}$$

Combining (4.1) with (4.2), we obtain

$$\int_{\mathbb{T}^n} \big\langle \big(\widetilde{R}_s(t) - \widetilde{G}_s(t)|_{J_s(t)}\big)\big(\mathcal{T}_s f(t)\big), \mathcal{T}_s f(t)\big\rangle_{\ell^2_s} \,\mathrm{d}t = 0, \quad f \in S_s(\mathcal{A}_{I,s}).$$

Thus, $\widetilde{R}_s(t) = \widetilde{G}_s(t)|_{J_s(t)} := R_s(t)$ for a.e. $t \in \mathbb{T}^n$. □

Theorem 4.2 ([17], **Theorem 5.2).** *Let $E_s(\mathcal{A}_{I,s})$ be a frame with constants A, B. Then, $E_s(\mathcal{B}_{I,s})$ is its dual frame with constants B^{-1}, A^{-1}, where $\mathcal{B}_{I,s} = \{\theta_i : \theta_i = L_0^{-1}\varphi_i, i \in I\}$. Furthermore,*

$$\mathcal{T}_s\theta_i(t) = R_s^{-1}(t)\big(\mathcal{T}_s\varphi_i(t)\big) \quad \text{for a.e. } t \in \mathbb{T}^n, i \in I. \tag{4.3}$$

Proof. Since

$$\begin{aligned}
\sum_{(k,i)\in\mathbb{Z}^n\times I} \big|\langle f, L_0^{-1}T_k\varphi_i\rangle_{H^s}\big|^2 &= \sum_{(k,i)\in\mathbb{Z}^n\times I} \big|\langle L_0^{-1}f, T_k\varphi_i\rangle_{H^s}\big|^2 \\
&= \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle L_0^{-1}f, T_k\varphi_i\rangle_{H^s}\overline{\langle L_0^{-1}f, T_k\varphi_i\rangle}_{H^s} \\
&= \sum_{(k,i)\in\mathbb{Z}^n\times I} \big\langle\langle L_0^{-1}f, T_k\varphi_i\rangle_{H^s}T_k\varphi_i, L_0^{-1}f\big\rangle_{H^s} \\
&= \Big\langle \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle L_0^{-1}f, T_k\varphi_i\rangle_{H^s}T_k\varphi_i, L_0^{-1}f\Big\rangle_{H^s} \\
&= \big\langle L_0(L_0^{-1}f), L_0^{-1}f\big\rangle_{H^s} = \langle L_0^{-1}f, f\rangle_{H^s},
\end{aligned}$$

we get

$$B^{-1}\|f\|_{H^s} \le \sum_{(k,i)\in\mathbb{Z}^n\times I} \big|\langle f, L_0^{-1}T_k\varphi_i\rangle_{H^s}\big|^2 = \langle L_0^{-1}f, f\rangle_{H^s} \le A^{-1}\|f\|_{H^s}. \tag{4.4}$$

Consequently, $\{L_0^{-1}T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is a dual frame of $E_s(\mathcal{A}_{I,s})$ with constants B^{-1}, A^{-1}. Furthermore, by Theorem 4.1, L_0 is a shift-preserving operator. Thus, for any $F = L_0 f \in H^s$, we have

$$L_0^{-1}T_kF = L_0^{-1}T_kL_0f = L_0^{-1}L_0T_kf = T_kf = T_kL_0^{-1}F, \quad k \in \mathbb{Z}^n.$$

Hence, L_0^{-1} is shift-preserving. Now, by (4.4), it follows that $E_s(\mathcal{B}_{I,s})$ is a dual frame of $E_s(\mathcal{A}_{I,s})$ with constants B^{-1}, A^{-1}. Moreover,

$$\sum_{(k,i)\in\mathbb{Z}^n\times I} \langle f, T_k\theta_i\rangle_{H^s}T_k\varphi_i = \sum_{(k,i)\in\mathbb{Z}^n\times I} \langle f, T_k\varphi_i\rangle_{H^s}T_k\theta_i = f \in S_s(\mathcal{A}_{I,s})$$

holds with the unconditional convergence in H^s.

Finally, (4.3) follows from (3.6) and Theorem 4.1. □

Remark 4.2. *Let $E_s(\mathcal{A}_{I,s})$ be a Riesz family with constants A, B. Then, the dual $E_s(\mathcal{B}_{I,s})$ is a Riesz family with constants B^{-1}, A^{-1}. Moreover,*

$$\langle T_k\varphi_i, T_\ell\theta_j\rangle_{H^s} = \delta_{k,\ell}\delta_{i,j}, \quad k,\ell \in \mathbb{Z}^n,\ i,j \in I,$$

where $\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$

5. Structural theorems

This section will be devoted to the main results of [7].

Recall that $\mathcal{D}_{L^2}(\mathbb{R}^n) = \bigcap_{s\geq 0} H^s$ and $\mathcal{D}'_{L^2}(\mathbb{R}^n) = \bigcup_{s\geq 0} H^{-s}$ (see for instance [28], [30]). Note that V_s is a closed subspace of H^s, $s \in \mathbb{R}$, so it is also a separable Hilbert space. The space of periodic L^2-functions, $L^2_{per}(\mathbb{R}^n)$, is defined by

$$L^2_{per}(\mathbb{R}^n) = \Big\{ g : g(\cdot) = \sum_{k\in\mathbb{Z}^n} a_k \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k\rangle},\ (a_k)_{k\in\mathbb{Z}^n} \in \ell^2 \Big\},$$

Theorem 5.1 ([7]). *Assume that $E_s(\mathcal{A}_{I,s})$ is a frame for V_s and that $E_s(\mathcal{B}_{I,s})$ is its dual frame, where $\mathcal{B}_{I,s} = \{\theta_i : \theta_i = L_0^{-1}\varphi_i, i \in I\}$. Then, $\mathcal{F}(V_s)$ is the set of Fourier transforms of elements $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ so that*

$$\widehat{f}(\cdot) = \sum_{i\in I} \widehat{\varphi}_i(\cdot) \sum_{k\in\mathbb{Z}^n} a^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k\rangle},$$

where $(a^i_k)_{k\in\mathbb{Z}^n} \in \ell^2$ is given by

$$a^i_k = \int_{\mathbb{R}^n} \widehat{f}(x) \mathrm{e}^{2\pi\sqrt{-1}\langle k, x\rangle} \overline{\widehat{\theta}_i}(x) \mu_s^2(x)\, \mathrm{d}x, \quad k \in \mathbb{Z}^n, i \in I. \tag{5.1}$$

Equivalently, it is equal to the space of elements $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ which Fourier transforms have the form

$$\widehat{f} = \sum_{i\in I} f_i g_i, \quad g_i \in L^2_{per}(\mathbb{R}^n),$$

where $f_i = \widehat{\varphi}_i \in L^2_s(\mathbb{R}^n)$, $i \in I$, and g_i, $i \in I$, have the expansions

$$g_i(\cdot) = \sum_{k\in\mathbb{Z}^n} a^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k\rangle},$$

with a^i_k determined by (5.1).

Instead of notation V^p_s in [26] (and V_p in [13]) we use $\mathcal{V}^2_s$, for $p = 2$. The main properties of the space $\mathcal{V}^2_s$ were announced in [26], where the weighted spaces V^p_s, $p \in [1, +\infty)$, were considered.

So, assume that $p = 2$. In the case $s = 0$, we assume that $\psi^i \in \mathcal{L}^\infty$, $i = 1, \dots, r$, where

$$\mathcal{L}^\infty = \Big\{ \psi : \|\psi\|_{\mathcal{L}^\infty} = \sup_{t\in\mathbb{T}^n} \sum_{j\in\mathbb{Z}^n} |\psi(t+j)| < +\infty \Big\}.$$

By [13], $\mathcal{V}^2 = \left\{ f : f = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^n} c^i_k T_k \psi^i,\ (c^i_k)_{k\in\mathbb{Z}^n} \in \ell^2,\ i = 1, \dots, r \right\}.$

Theorem 5.2 ([7]). *Assume that $\mathcal{A}_r = \{\psi^i :\ i = 1, \dots, r\} \subset L^2(\mathbb{R}^n) \cap \mathcal{L}^\infty$. Then,*

$$\mathcal{V}^2 = V_0 = S_0(\mathcal{A}_r),$$

if $\mathcal{V}^2$ is closed in $L^2(\mathbb{R}^n)$.

Now, we consider weighted versions. Let $s > 0$ be fixed. We will introduce several assumptions on generators ψ^i, $i = 1, \dots, r$, in order to have that their linear combinations determine subspaces of H^s and of L^2_s:

$$\psi^i \in H^s \cap L^2_s, \quad i = 1, \dots, r. \tag{5.2}$$

Moreover, in order to have the same assumptions as in [13] (and [26]), we assume, as in the previous assumption, that

$$\psi^i \in \mathcal{L}^\infty, \quad i = 1, \dots, r. \tag{5.3}$$

Let

$$\mathcal{V}^2_s = \left\{ f : f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} c^i_k T_k \psi^i, \ (c^i_k)_{k \in \mathbb{Z}^n} \in \ell^2_s, \ i = 1, \dots, r \right\}, \tag{5.4}$$

by recalling the notation in [26].

Theorem 5.3 ([7]). *Assume that $s > 0$, (5.2) and (5.3) hold.*

(1) *Assume that*

$$\mathcal{V}_s \text{ and } \mathcal{F}(\mathcal{V}^2_s) \text{ are closed in } L^2_s.$$

Then,

$$\mathcal{V}^2_s \subset H^s \text{ and } \mathcal{V}^2_s = V_s = S_s(\mathcal{A}_{r,s}).$$

In particular, any element $f \in V_s$ has the frame expansion as in (5.4).

(2) *Assume that $s > 1/2$ and that $\mathcal{V}^2_s$ is closed in L^2_s. Then, $\mathcal{F}(\mathcal{V}^2_s)$ is closed in L^2_s and both assertions in (1) hold true.*

Concerning the duality, we have the following assertion.

Theorem 5.4 ([7]). *Assume that $s > 0$, (5.2) and (5.3) hold. Moreover, assume that the conditions of assertion (1) or conditions of assertion (2) of Theorem 5.3 hold. Then in (both cases),*

(1) $(\mathcal{V}^2_s)' = \mathcal{V}^2_{-s}$, *where $\mathcal{V}^2_{-s}$ is the space of formal series of the form*

$$F(\cdot) = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} b^i_k \psi^i(\cdot - k), \quad \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} |b^i_k|^2 (1 + |k|^2)^{-s} < +\infty,$$

with the dual pairing

$$\langle F, f \rangle = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} b^i_k c^i_k, \ \ (f \text{ is of the form given in } (5.4)).$$

(2) $\mathcal{V}^2_{-s} = V_{-s}$.

In order to consider the intersections of V_s, $s \geq 0$, instead of conditions (5.2) and (5.3), we assume

$$\psi^i \in \mathcal{S}(\mathbb{R}^n), \quad i = 1, \dots, r. \tag{5.5}$$

Theorem 5.5 ([7]). *Assume that* (5.5) *holds. Then,*

$$\bigcap_{s\geq 0}\mathcal{V}_s^2=\bigcap_{s\geq 0}V_s,$$

and the expansion for their elements has the form as in (5.4) *with*

$$\sup_{k\in\mathbb{Z}^n}|c_k^i|k^s<+\infty,\quad i=1,\dots,r,\ \textit{for every } s>0.$$

Recall that the space $\mathcal{P}(\mathbb{R}^n)=\mathcal{P}$ of periodic smooth test functions (with period one in any variable) is given by

$$\mathcal{P}=\Big\{\phi:\phi(\cdot)=\sum_{k\in\mathbb{Z}^n}a_k\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle},\ (a_k)_{k\in\mathbb{Z}^n}\in\ell_s^2 \text{ for every } s\geq 0\Big\},$$

while its dual space $\mathcal{P}'(\mathbb{R}^n)=\mathcal{P}'$ is given by

$$\mathcal{P}'=\Big\{\phi:\phi(\cdot)=\sum_{k\in\mathbb{Z}^n}a_k\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle},\ (a_k)_{k\in\mathbb{Z}^n}\in\ell_{-s}^2 \text{ for some } \ s\geq 0\Big\}.$$

A direct consequence part (2) of Theorem 5.3 is the following assertion.

Corollary 5.1 ([7]). *Assume that* (5.5) *holds. Then*

$$\mathcal{F}\Big(\bigcap_{s\geq 0}\mathcal{V}_s^2\Big)$$
$$=\Big\{\sum_{i=1}^r\widehat{\psi}^i(\cdot)\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}:(c_k^i)_{k\in\mathbb{Z}^n}\in\ell_s^2, i=1,\dots,r,\ \textit{for every}\ \ s\geq 0\Big\},$$

where $\Phi_i(\cdot)=\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\in\mathcal{P}$, $i=1,\dots,r$.

Concerning the duality, by Theorem 5.4, we have the next corollary.

Corollary 5.2 ([7]). *Assume that* (5.5) *holds. Then* $V_s'=\mathcal{V}_{-s}^2$, $\bigcup_{s>0}V_s'=\bigcup_{s>0}\mathcal{V}_{-s}^2$ *and*

$$\mathcal{F}\Big(\bigcup_{s\leq 0}\mathcal{V}_s^2\Big)$$
$$=\Big\{\sum_{i=1}^r\widehat{\psi}^i(\cdot)\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}:(c_k^i)_{k\in\mathbb{Z}^n}\in\ell_s^2, i=1,\dots,r, \textit{for some } s\leq 0\Big\},$$

where $F_i(\cdot)=\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\in\mathcal{P}'$, $i=1,\dots,r$.

Note that the assumption $\psi^i\in\mathcal{S}(\mathbb{R}^n)$ implies a well defined product of a smooth function and a (periodic) Schwartz distribution.

6. Spectral analysis of the range operator

The present section contains relevant concepts and auxiliary results needed to prove theorems in the last section. These specialize to the results of [5] if $s = 0$. Ideas used in proofs are in much the same way as in [5].

Theorem 6.1 ([5])**.** *Let $A = A(t)$ be an $n \times n$ matrix of measurable functions defined on a measurable set B. Then, there exist n measurable functions $\lambda_i : B \to \mathbb{C}$, $i = 1, \dots, n$, such that $\lambda_1(t), \dots, \lambda_n(t)$ are eigenvalues for matrix $A(t)$, for a.e. $t \in B$.*

Theorem 6.2 ([5], **Proposition 3.4, Theorem 5.4**)**.** *Let J_s be a range function for the shift-invariant space $V_s \subset H^s$ such that $\dim J_s(t) = d < +\infty$ for a.e. $t \in \Omega$, where $\Omega \subseteq \mathbb{T}^n$ is measurable set, and let $L : V_s \to V_s$ be a shift-preserving operator with the corresponding range operator R_s. Then, there exist d^2 measurable bounded functions $(R_s^{ij})_{i,j=1}^d$ defined on Ω, such that*

$$[R_s(t)] = \begin{bmatrix} R_s^{11}(t) & R_s^{12}(t) & \cdots & R_s^{1d}(t) \\ R_s^{21}(t) & R_s^{22}(t) & \cdots & R_s^{2d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ R_s^{d1}(t) & R_s^{d2}(t) & \cdots & R_s^{dd}(t) \end{bmatrix},$$

for a.e. $t \in \Omega$. Moreover, there exist d measurable functions $\lambda_s^i : \Omega \to \mathbb{C}$, $i = 1, \dots, d$, so that $\lambda_s^1(t), \dots, \lambda_s^d(t)$ are eigenvalues for $R_s(t)$ for a.e. $t \in \Omega$.

Proof. Let us consider $(A_d)_{d \in \mathbb{N}_0}$ and $\varphi_i \in H^s$, $i \in \mathbb{N}$, from Proposition 2.2. Since $\{\mathcal{T}_s\varphi_1(t), \dots, \mathcal{T}_s\varphi_d(t)\}$ is the orthonormal basis for $J_s(t)$ for a.e. $t \in A_d$, we can conclude that $R_s(t)$ has a matrix representation with elements

$$R_s^{ij}(t) := \big\langle R_s(t)\mathcal{T}_s\varphi_j(t), \mathcal{T}_s\varphi_i(t)\big\rangle_{\ell_s^2},$$

for a.e. $t \in A_d$. Obviously, $R_s^{ij}(t)$ is measurable for a.e. $t \in A_d$. Since L is bounded, by Theorem 3.2, we have $|R_s^{ij}(t)| \leq \|L\|$, $i, j = 1, \dots, d$, for a.e. $t \in A_d$. Clearly, the set Ω is included in A_d.

Let $\nu_s(t) : J_s(t) \to \mathbb{C}^d$ be given by $\nu_s(t)\big(\mathcal{T}_s\varphi_i(t)\big) = \mathrm{e}_i$, for a.e. $t \in A_d$, $i = 1, \dots, d$, where $\{\mathrm{e}_1, \dots, \mathrm{e}_d\}$ is the canonical basis for $\mathbb{C}^d$. In this way, the unique connection between bases is established, for a.e. $t \in A_d$. Then, $R_s(t) = \nu_s(t)^{-1}[R_s(t)]\nu_s(t)$, for a.e. $t \in \Omega$. For a measurable function $\lambda_s : \Omega \to \mathbb{C}$ holds

$$\ker\big(R_s(t) - \lambda_s(t)1(t)\big) = \ker\big(\nu_s(t)^{-1}([R_s(t)] - \lambda_s(t)1(t))\nu_s(t)\big), \quad \text{a.e. } t \in \Omega.$$

Therefore, by Theorem 6.1, the assertion follows. □

For a bounded measurable range operator R_s, we will denote by $\mathfrak{R}_\lambda(t)$ the set of all eigenvalues of $R_s(t)$. The length of FSI space $V_s \subset H^s$, denoted by $\Gamma(V_s)$, is defined to be the smallest $r \in \mathbb{N}$ such that $V_s = S_s(\varphi_1, \dots, \varphi_r)$, where $\varphi_1, \dots, \varphi_r \in V_s$. In the notation of [5], the equivalent definition is $\Gamma(V_s) = \operatorname{ess\,sup}_{t \in \mathbb{T}^n} \dim J_s(t)$, where J_s corresponds to V_s.

Theorem 6.3 ([5], Theorem 5.5). *Let J_s be a range function so that* $\dim J_s(t) < +\infty$ *for a.e.* $t \in \mathbb{T}^n$*, and let* $R_s(t) : J_s(t) \to J_s(t)$ *be a bounded measurable range operator. Then, there exist* $\lambda_s^i \in L^\infty(\mathbb{T}^n)$*,* $i \in \mathbb{N}$*, such that* $\lambda_s^i(t) \neq \lambda_s^j(t)$*,* $i \neq j$*, for a.e.* $t \in \mathbb{T}^n$*, and if* $A_{d,q} := \{t \in A_d : \operatorname{card}(\mathfrak{R}_\lambda(t)) = q\}$*, where the sequence* $(A_d)_{d\in\mathbb{N}_0}$ *is given by Proposition* 2.2*, then* $\mathfrak{R}_\lambda(t) = \{\lambda_s^1(t), \dots, \lambda_s^q(t)\}$ *for a.e.* $t \in A_{d,q}$*, for all* $q \leq d$ *and* $d, q \in \mathbb{N}$*.*

Proof. By Proposition 2.2, $A_i \cap A_j = \emptyset$, $i \neq j$, and $\sigma_{V_s} = \bigcup_{d\in\mathbb{N}} A_d$. Moreover, by Theorem 6.2, for every $d \in \mathbb{N}$ there exist d measurable functions $\lambda_s^{d,1}, \dots, \lambda_s^{d,d} : A_d \to \mathbb{C}$ which are the eigenvalues of $R_s(t)$ for a.e. $t \in A_d$. Let us regard $d \in \mathbb{N}$ as fixed and, for all $q \leq d$, define

$$A_{d,q} := \big\{t \in A_d : \operatorname{card}\{\lambda_s^{d,1}(t), \dots, \lambda_s^{d,d}(t)\} = q\big\}.$$

These sets are measurable and disjoint and $A_d = \bigcup_{q=1}^d A_{d,q}$. Now, there exist measurable functions $\lambda_s^{d,q;1}, \dots, \lambda_s^{d,q;q} : A_{d,q} \to \mathbb{C}$ which are eigenvalues of $R_s(t)$ such that $\lambda_s^{d,q;i}(t) \neq \lambda_s^{d,q;j}(t)$, $i \neq j$, for a.e. $t \in A_{d,q}$. Further, since R_s is bounded, there exists a constant $C > 0$ such that $|\lambda_s^{d,q;i}(t)| \leq C$ for a.e. $t \in A_{d,q}$ and $i \leq q \leq d$. Let us define the functions $\lambda_s^i : \mathbb{T}^n \to \mathbb{C}$ by

$$\lambda_s^i(t) := \begin{cases} \lambda_s^{d,q;i}(t), & t \in A_{d,q},\ i \leq q \leq d, \\ C + i, & \text{otherwise.} \end{cases}$$

Then, $\lambda_s^i(t) \neq \lambda_s^j(t)$, $i \neq j$, for a.e. $t \in \mathbb{T}^n$ and $\lambda_s^i \in L^\infty(\mathbb{T}^n)$, $i \in \mathbb{N}$. Moreover, $\lambda_s^i(t)$ is the eigenvalue of $R_s(t)$ for a.e. $t \in A_{d,q}$, since

$$\ker\big(R_s(t) - \lambda_s^i(t)1(t)\big) = \ker\big(R_s(t) - \lambda_s^{d,q;i}(t)1(t)\big),$$

for a.e. $t \in A_{d,q}$ and all $i \leq q \leq d$. Otherwise, since $\lambda_s^i(t) = C + i$ is not the eigenvalue of $R_s(t)$, we have $\ker\big(R_s(t) - \lambda_s^i(t)1(t)\big) = \{\mathbf{0}\}$. ☐

Remark 6.1. (1) *If J_s is the range function for FSI space $V_s \subset H^s$, then the measure of the set A_d is* 0 *for every* $d > \Gamma(V_s)$ *(we write it* $|A_d| = 0$*,* $d > \Gamma(V_s)$*, for short). Thus, for* $q \in \mathbb{N}$*, we define*

$$B_q := \bigcup_{d=q}^{+\infty} A_{d,q}, \quad \text{and} \quad \mathbf{q} := \max\{q \in \mathbb{N} : |B_q| \neq 0\}. \tag{6.1}$$

(2) *Let us define* $C_i := \bigcup_{q=i}^{+\infty} B_q$*,* $i \in \mathbb{N}$*. The set* C_i *contains* $t \in \sigma_{V_s}$ *such that the range operator* $R_s(t)$ *has at least i different eigenvalues. By the proof of Theorem* 6.3*, it is clear that* $C_i = \big\{t \in \mathbb{T}^n : \ker(R_s(t) - \lambda_s^i(t)1(t)) \neq \{\mathbf{0}\}\big\}$*,* $i \in \mathbb{N}$*, and* $|C_i| = 0$ *for* $i > \mathbf{q}$*.*

7. s-Diagonalization and dynamical sampling for shift-preserving operators

The definition of s-diagonalization was first introduced by Aguilera et al. in [5]. In this section, we follow the notation used in [5] and [6] and adapt the definition for the space H^s and give conditions under which the shift-preserving operator will be s-diagonalizable. In the last part of this section, we show that iterations of the shift-preserving operator on a finite set of functions produce a frame for V_s.

Let $\psi : \mathbb{T}^n :\to \mathbb{C}$ be measurable and M_ψ be a multiplication operator $M_\psi : H(\mathbb{T}^n, \ell_s^2) \to H(\mathbb{T}^n, \ell_s^2)$,

$$M_\psi \mathcal{T}_s f(t) = \left(\frac{\psi(t)\widehat{g}(t+k)}{\mu_s(k)} \right)_{k\in\mathbb{Z}^n}, \quad t \in \mathbb{T}^n,$$

where $g = (1 - \frac{\Delta}{4\pi^2})^{s/2} f$. It is continuous if and only if $\psi \in L^\infty(\mathbb{T}^n)$.

For a sequence $a = (a_k)_{k\in\mathbb{Z}^n} \in \ell_s^2$, we define

$$\widehat{a}(\cdot) = \sum_{k\in\mathbb{Z}^n} a_k \mathrm{e}^{-2\pi\sqrt{-1}\langle k,\cdot\rangle}$$

and an operator $\Lambda_a^s : H^s \to H^s$ by $\Lambda_a^s = \sum_{k\in\mathbb{Z}^n} a_k T_k$. If $\widehat{a}(\cdot) \in L^\infty(\mathbb{T}^n)$, then the sequence $a = (a_k)_{k\in\mathbb{Z}^n} \in \ell_s^2$ is called a sequence with bounded spectrum. We say that Λ_a^s is an s-eigenvalue of L if $V_s^a := \{f \in H^s : Lf = \Lambda_a^s f\} \neq \{\mathbf{0}\}$, where $a = (a_k)_{k\in\mathbb{Z}^n} \in \ell_s^2$ is a sequence with bounded spectrum. The space V_s^a is called s-eigenspace associated to Λ_a^s. Note that 's' in 's-eigenvalue' and 's-eigenspace' has nothing to do with 's' in the space H^s, even though we use the same letter.

The following lemma will prove extremely useful.

Lemma 7.1. *Let $a = (a_k)_{k\in\mathbb{Z}^n} \in \ell_s^2$. Then $\Lambda_a^s = \mathcal{T}_s^{-1} M_{\widehat{a}}^s \mathcal{T}_s : H^s \to H^s$ is linear bounded operator if and only if a has a bounded spectrum.*

Proof. Let $f \in H^s$, then $(1 - \Delta/(4\pi^2))^{s/2} f =: g \in L^2$ and

$$\begin{aligned}
\mathcal{T}_s^{-1} M_{\widehat{a}}^s \mathcal{T}_s f(t) &= \mathcal{T}_s^{-1} \left(\frac{\widehat{a}(t)\widehat{g}(t+j)}{\mu_s(j)} \right)_{j\in\mathbb{Z}^n} \\
&= \sum_{k\in\mathbb{Z}^n} a_k \mathcal{T}_s^{-1} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \left(\frac{\widehat{g}(t+j)}{\mu_s(j)} \right)_{j\in\mathbb{Z}^n} \\
&= \sum_{k\in\mathbb{Z}^n} a_k \mathcal{T}_s^{-1} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \mathcal{T}_s f(t) = \sum_{k\in\mathbb{Z}^n} a_k T_k f(t) = \Lambda_a^s f(t),
\end{aligned}$$

where we used Lemma 2.1. Hence, $\Lambda_a^s = \mathcal{T}_s^{-1} M_{\widehat{a}}^s \mathcal{T}_s$. Linearity simply follows from the definition of operator Λ_a^s. Since $\mathcal{T}_s$ is an isometric isomorphism (see Lemma 2.1), the statement follows. □

Next tree lemmas are necessary for Theorem 7.1.

Lemma 7.2. *For $f \in V_s^a$, we have $R_s(t)\big(\mathcal{T}_s f(t)\big) = \widehat{a}(t)\mathcal{T}_s f(t)$, for a.e. $t \in \mathbb{T}^n$.*

Proof. Using (3.6), Lemma 7.1 and $Lf = \Lambda_a^s f$, we have

$$R_s(t)\big(\mathcal{T}_s f(t)\big) = \mathcal{T}_s\big(Lf\big)(t) = \mathcal{T}_s\big(\Lambda_a^s f\big)(t) = \mathcal{T}_s\big(\mathcal{T}_s^{-1} M_{\widehat{a}}^s \mathcal{T}_s f\big)(t) = \widehat{a}(t)\mathcal{T}_s f(t),$$

for a.e. $t \in \mathbb{T}^n$. □

Lemma 7.3. *Let $V_s \subset H^s$ be a shift-invariant space. Then, there exists $\varphi \in V_s$ such that* $\operatorname{supp} \|\mathcal{T}_s\varphi(\cdot)\|_{\ell^2_s} = \sigma_{V_s}$.

Proof. The proof is a direct consequence of [15, Proposition 2.9] and therefore is omitted. □

The next lemma may be proved in much the same way as [5, Proposition 3.5], so the proof is omitted.

Lemma 7.4. *Let J_s be a range function such that $\dim J_s(t) < +\infty$ and let $R_s(t) : J_s(t) \to J_s(t)$ be a measurable range operator for a.e. $t \in \mathbb{T}^n$. Then, the mapping $t \mapsto \ker(R_s(t))$, $t \in \mathbb{T}^n$, is a measurable range function.*

Using the lemmas 7.2, 7.3 and 7.4, we get the following statement.

Theorem 7.1 ([5], **Proposition 4.5**). *Let J_s be a range function for shift-invariant space $V_s \subset H^s$ such that $\dim J_s(t) < +\infty$ for a.e. $t \in \mathbb{T}^n$. Let $a \in \ell^2_s$ be a sequence of bounded spectrum and let $L : V_s \to V_s$ be a bounded shift-preserving operator. If Λ^s_a is an s-eigenvalue of L, then $\Lambda^s_a(t) := \widehat{a}(t)$ is an eigenvalue of $R_s(t)$ for a.e. $t \in \sigma_{V^a_s}$. Moreover, $J^a_s(t) := \ker\big(R_s(t) - \Lambda^s_a(t)1(t)\big)$, for a.e. $t \in \mathbb{T}^n$, is the measurable range function for V^a_s.*

Remark 7.1. *From the previous, it follows that:*

(1) *V^a_s is a shift-invariant subspace of V_s,*
(2) *$LV^a_s \subseteq V^a_s$,*
(3) *if $V^a_s \cap V^b_s = \{\mathbf{0}\}$, then $\widehat{a}(t) \neq \widehat{b}(t)$ a.e. in $\sigma_{V^a_s} \cap \sigma_{V^b_s}$.*

In the remainder of this section, we assume $V_s \subset H^s$ to be FSI space with the range function J_s and $L : V_s \to V_s$ to be a shift-preserving operator with the corresponding range operator R_s.

We will say that $(V_s, L, a^1, \dots, a^p)$ is an s-diagonalization of L, i.e. the operator L is s-diagonalizable, if there exists natural number p so that $\Lambda^s_{a^i}$, $i = 1, \dots, p$, are s-eigenvalues of L and $V_s = V^{a^1}_s \oplus V^{a^2}_s \oplus \cdots \oplus V^{a^p}_s$, where $\oplus$ denote direct sum and a^i, $i = 1, \dots, p$, are sequences of bounded spectrum. Note that 's' in 's-diagonalization' has nothing to do with 's' in the space H^s, even though we use the same letter.

Theorem 7.2 ([5], **Theorem 6.4**). *If L is s-diagonalizable, then $R_s(t)$ is diagonalizable for a.e. $t \in \sigma_{V_s}$.*

Proof. Assume that $(V_s, L, a^1, \dots, a^p)$ is an s-diagonalization of L. Then, by Theorem 7.1, $\lambda^s_{a^\ell}(t) := \widehat{a}^\ell(t)$ is an eigenvalue of $R_s(t)$ for a.e. $t \in \sigma_{V^{a^\ell}_s}$ and $J^{a^\ell}_s(t) := \ker\big(R_s(t) - \widehat{a}^\ell(t)1(t)\big)$ is eigenspace for $R_s(t)$, for a.e. $t \in \mathbb{T}^n$, for $\ell = 1, \dots, p$. We only need to show that $J_s(t) = J^{a^1}_s(t) \oplus \cdots \oplus J^{a^p}_s(t)$, for a.e. $t \in \mathbb{T}^n$.

We check at once that $J^{a^1}_s(t) + \cdots + J^{a^p}_s(t) \subseteq J_s(t)$, which is clear from

$$V^{a^\ell}_s \subseteq V_s \;\Rightarrow\; J^{a^\ell}_s(t) \subseteq J_s(t), \quad \ell = 1, \dots, p.$$

For $\phi \in V_s$, we have $\phi = \phi_1 + \cdots + \phi_p$, where $\phi_\ell \in V_s^{a^\ell}$, $\ell = 1, \dots, p$. Thus, $\mathcal{T}_s\phi(t) = \mathcal{T}_s\phi_1(t) + \cdots + \mathcal{T}_s\phi_p(t)$ for a.e. $t \in \mathbb{T}^n$, and therefore

$$J_s(t) = \overline{\operatorname{span}}\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{r,s}\} \subseteq \overline{J_s^{a^1}(t) + \cdots + J_s^{a^p}(t)} = J_s^{a^1}(t) + \cdots + J_s^{a^p}(t),$$

where $V_s = S_s(\mathcal{A}_{r,s})$. The second equality follows from the fact that $\dim J_s(t) < +\infty$ for a.e. $t \in \mathbb{T}^n$.

From the above, we conclude that $J_s(t) = J_s^{a^1}(t) + \cdots + J_s^{a^p}(t)$, for a.e. $t \in \mathbb{T}^n$. Using Proposition 2.1, we get $J_s(t) = J_s^{a^1}(t) \oplus \cdots \oplus J_s^{a^p}(t)$, for a.e. $t \in \mathbb{T}^n$, i.e. the sum is direct. □

Theorem 7.3 ([5], **Theorem 6.8**). *If the operator $R_s(t)$ is diagonalizable for a.e. $t \in \sigma_{V_s}$, then there exist sequences $(a^i)_{i=1}^{\mathbf{q}}$ of bounded spectrum, where $\mathbf{q}$ is given by* (6.1)*, such that $J_s^{a^i}(t) := \ker\big(R_s(t) - \widehat{a}^i(t)1(t)\big)$, $i = 1, \dots, \mathbf{q}$, are measurable range functions and*

1. *$J_s(t) = J_s^{a^1}(t) \oplus \cdots \oplus J_s^{a^{\mathbf{q}}}(t)$, for a.e. $t \in \mathbb{T}^n$, where $\oplus$ denote direct sum;*
2. *the sets $C_i = \big\{t \in \sigma_{V_s} : J_s^{a^i}(t) \neq \{\mathbf{0}\}\big\}$, $i = 1, \dots, \mathbf{q}$, satisfy $|C_i| > 0$ and $C_{i+1} \subset C_i$, $i = 1, \dots, \mathbf{q} - 1$.*

Proof. (1) Let $\Gamma(V_s) = r$. By Theorem 6.3 and Remark 6.1, there exist measurable functions $\lambda_s^1, \dots, \lambda_s^{\mathbf{q}} \in L^\infty(\mathbb{T}^n)$ such that

$$J_s(t) = \bigoplus_{i=1}^{\mathbf{q}} \ker\big(R_s(t) - \lambda_s^i(t)1(t)\big), \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Indeed, if $t \in A_{d,q}$, then (see the proof of Theorem 6.3)

$$\bigoplus_{i=1}^{\mathbf{q}} \ker\big(R_s(t) - \lambda_s^i(t)1(t)\big) = \bigoplus_{i=1}^{q} \ker\big(R_s(t) - \lambda_s^{d,q;i}(t)1(t)\big) \oplus \bigoplus_{i=q+1}^{\mathbf{q}} \{\mathbf{0}\} = J_s(t),$$

since $R_s(t)$ is diagonalizable and $\{\lambda_s^{d,q;1}(t), \dots, \lambda_s^{d,q;q}(t)\}$ is the set of eigenvalues of $R_s(t)$ on $A_{d,q}$, for a.e. $t \in \mathbb{T}^n$. If $t \notin \sigma_{V_s}$, then $\ker\big(R_s(t) - \lambda_s^i(t)1(t)\big) = \{\mathbf{0}\}$, $i = 1, \dots, \mathbf{q}$, and $J_s(t) = \{\mathbf{0}\}$.

Moreover, since $\lambda_s^i \in L^\infty(\mathbb{T}^n)$, there exists a sequence $a^i = (a_k^i)_{k \in \mathbb{Z}^n} \in \ell_s^2$ of bounded spectrum so that $\lambda_s^i(t) = \widehat{a}^i(t)$ for a.e. $t \in \mathbb{T}^n$ and the function $J_s^{a^i}(t) = \ker\big(R_s(t) - \widehat{a}^i(t)1(t)\big)$ is measurable. Hence, (1) holds.

(2) It follows from Remark 6.1(2). □

Now we can prove a theorem which gives necessary conditions for s-diagonalization of shift-preserving operator L. The main idea of the proof is presented in [5].

Theorem 7.4 ([5], **Theorem 6.16**). *If $L : V_s \to V_s$ is a normal shift-preserving operator, then L is s-diagonalizable and*

$$L = \sum_{i=1}^{p} \Lambda_{a^i}^s P_{V_s^{a^i}},$$

where $(V_s, L, a^1, \dots, a^p)$ *is an* s*-diagonalization of* L *and* $P_{V_s^{a^i}} : V_s \to V_s^{a^i}$, $i = 1, \dots, p$, *are the orthogonal projections.*

Proof. Since L is normal and V_s is a FSI space, by Theorem 3.4, the range operator $R_s(t)$ is normal for a.e. $t \in \mathbb{T}^n$, and therefore the range operator $R_s(t)$ is diagonalizable for a.e. $t \in \mathbb{T}^n$ and moreover, its eigenspaces are orthogonal.

Let $\mathbf{q}$ be given by (6.1). By Theorem 7.3, we have $J_s(t) = J_s^{a^1}(t) \oplus \cdots \oplus J_s^{a^{\mathbf{q}}}(t)$ for a.e. $t \in \mathbb{T}^n$, where $J_s^{a^i}(t) = \ker\big(R_s(t) - \widehat{a}^i(t)1(t)\big)$ is the measurable range function and a^i is the sequence of bounded spectrum for all $i = 1, \dots, \mathbf{q}$.

Using Theorem 7.1, we get that $V_s^{a^i} = \{f \in H^s : Lf = \Lambda^s_{a^i} f\} \neq \{\mathbf{0}\}$ is the shift-invariant space with the measurable range function $J_s^{a^i}$. The orthogonality between the eigenspaces $J_s^{a^i}(t)$, $i = 1, \dots, \mathbf{q}$, for a.e. $t \in \mathbb{T}^n$, implies the orthogonality between the s-eigenspaces $V_s^{a^i}$, $i = 1, \dots, \mathbf{q}$. Finally, by Proposition 2.1(2), we get $V_s = V_s^{a^1} \oplus \cdots \oplus V_s^{a^{\mathbf{q}}}$ and consequently, the shift-preserving operator L is s-diagonalizable.

If $(V_s, L, a^1, \dots, a^p)$ is any s-diagonalization for L, since the eigenspaces of $R_s(t)$ are orthogonal for a.e. $t \in \mathbb{T}^n$, we get that the s-eigenspaces $V_s^{a^i}$ are orthogonal and $V_s = V_s^{a^1} \oplus \cdots \oplus V_s^{a^p}$. Thus, the assertion holds. □

Theorem 7.5 ([6], Proposition 2.16). *If the operator* $L : V_s \to V_s$ *is normal, then the following assertions hold.*

(1) *The operators* L *and its adjoint* L^* *are* s*-diagonalizable.*
(2) *If the operator* Λ^s_a *is an* s*-eigenvalue of operator* L*, then its adjoint, denoted by* Λ^{s*}_a*, is an* s*-eigenvalue of* L^* *and* $V_s^a = \{f \in H^s : L^* f = \Lambda^{s*}_a f\}$*. Moreover,* $\Lambda^{s*}_a = \Lambda^s_{\widetilde{a}}$*, where* $\widetilde{a} \in \ell^2_s$ *and* $\widetilde{a}_k := \overline{a}_{-k}$*,* $k \in \mathbb{Z}^n$*.*
(3) *If* $(V_s, L, a^1, \dots, a^p)$ *is an* s*-diagonalization for operator* L*, then* $(V_s, L^*, \widetilde{a}^1, \dots, \widetilde{a}^p)$ *is an* s*-diagonalization for* L^**.*

Proof. (1) The adjoint L^* of normal operator L is also normal, and thus, by Theorem 7.4, L and L^* are s-diagonalizable.

(2) Since

$$\overline{\widehat{a}}(t) = \sum_{k \in \mathbb{Z}^n} \overline{a}_k \mathrm{e}^{2\pi\sqrt{-1}\langle k,t\rangle} = \sum_{k \in \mathbb{Z}^n} \overline{a}_{-k} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle},$$

write $\widetilde{a}_k := \overline{a}_{-k}$, $k \in \mathbb{Z}^n$. It is clear that $\widetilde{a} = (\widetilde{a}_k)_{k \in \mathbb{Z}^n} \in \ell^2_s$. For Λ^s_a which is an s-eigenvalue of L and $\varphi_1, \varphi_2 \in V_s$, we get

$$\langle \Lambda^s_a \varphi_1, \varphi_2 \rangle_{H^s} = \langle \varphi_1, \Lambda^s_{\widetilde{a}} \varphi_2 \rangle_{H^s},$$

i.e. $\Lambda^{s*}_a = \Lambda^s_{\widetilde{a}}$. Moreover, since $L - \Lambda^s_a$ is a normal operator, we get $\ker(L^* - \Lambda^s_{\widetilde{a}}) = \ker((L - \Lambda^s_a)^*) = \ker(L - \Lambda^s_a) = V_s^a \neq \{\mathbf{0}\}$, and the assertion holds.

(3) Let $(V_s, L, a^1, \dots, a^p)$ be an s-diagonalization for L. By (2), we have $\ker(L^* - \Lambda^s_{\widetilde{a}^j}) = V_s^{a^j}$, $j = 1, \dots, p$, and therefore $V_s = V_s^{a^1} \oplus \cdots \oplus V_s^{a^p}$ is decomposition on s-eigenspaces for L^*. □

We introduce the notion of $M = \{1, 2, \ldots, m\}$, $D = \{0, 1, \ldots, d-1\}$ and $V_s^b := \{f \in H^s : L^* f = \Lambda_b^s f\}$, where L^* is the adjoint operator of shift-preserving operator $L : V_s \to V_s$ and $b \in \ell_s^2$ is a sequence of bounded spectrum such that Λ_b^s is an s-eigenvalue of L^*. By Theorem 7.5, if L is normal, then $\Lambda_b^s = \Lambda_{\bar{a}}^s$, where Λ_a^s is an s-eigenvalue of L.

The following assertion is a restatement of [6, Theorem 3.2].

Theorem 7.6. *Let $V_s = S_s(\varphi_1, \ldots, \varphi_d) \subset H^s$ be a shift-invariant space and let $J_s^b(t) := \ker\big(R_s^*(t) - \widehat{b}(t)1(t)\big)$, where $\widehat{b}(t)$ is eigenvalue of $R_s^*(t)$ for a.e. $t \in \sigma_{V_s^b}$. If $\{R_s^j(t)(\mathcal{T}_s\phi_i(t)) : \phi_i \in V_s, i \in M, j \in D\}$ is a frame for $J_s(t)$ with a frame bounds $A, B > 0$, for a.e. $t \in \sigma_{V_s^b}$, then $\{P_{J_s^b}(t)(\mathcal{T}_s\phi_i(t)) : \phi_i \in V_s, i \in M\}$ is a frame for $J_s^b(t)$ with bounds $A/C(t), B/C(t)$, for a.e. $t \in \sigma_{V_s^b}$, where $C(t) = \sum_{j\in D} |\widehat{b}(t)|^{2j}$.*

Proof. The assertion follows from the next equalities.

$$\begin{aligned}\sum_{j\in D}\sum_{i\in M}\left|\left\langle \mathcal{T}_s\phi(t), R_s^j(t)(\mathcal{T}_s\phi_i(t))\right\rangle_{\ell_s^2}\right|^2 &= \sum_{j\in D}\sum_{i\in M}\left|\left\langle R_s^{*^j}(t)(\mathcal{T}_s\phi(t)), \mathcal{T}_s\phi_i(t)\right\rangle_{\ell_s^2}\right|^2 \\ &= \sum_{j\in D}\sum_{i\in M}\left|\left\langle \widehat{b}(t)^j(\mathcal{T}_s\phi(t)), \mathcal{T}_s\phi_i(t)\right\rangle_{\ell_s^2}\right|^2 \\ &= \sum_{j\in D}|\widehat{b}(t)|^{2j}\sum_{i\in M}\left|\left\langle \mathcal{T}_s\phi(t), P_{J_s^b}(t)(\mathcal{T}_s\phi_i(t))\right\rangle_{\ell_s^2}\right|^2,\end{aligned}$$

where $\widehat{b}(t)$ is eigenvalue of $R_s^*(t)$, for a.e. $t \in \sigma_{V_s^b}$. □

Now, we can prove our main result.

Theorem 7.7 ([6], Theorem 3.6). *Let $V_s = S_s(\varphi_1, \ldots, \varphi_d) \subset H^s$ be a shift-invariant space and let $L : V_s \to V_s$ be a shift-preserving operator with the corresponding range operator R_s. If*

$$\{L^j\phi_i : \phi_i \in V_s, i \in M, j \in D\}$$

is a frame generator set for V_s with bounds $A, B > 0$, then

$$\{P_{V_s^b}\phi_i : \phi_i \in V_s, i \in M\}$$

is a frame generator set for V_s^b with bounds $A\big(\sum_{j=0}^{d-1}\|L\|^{2j}\big)^{-1}$ and B.

Proof. Assume that $\{L^j\phi_i : \phi_i \in V_s, i \in M, j \in D\}$ is a frame generator set for V_s with bounds $A, B > 0$. Using Theorem 2.2 and the equality (3.6), we get

$$\{\mathcal{T}_s\big(L^j\phi_i\big)(t) : i \in M, j \in D\} = \{R_s^j(t)\big(\mathcal{T}_s\phi_i(t)\big) : i \in M, j \in D\}$$

is a frame for $J_s(t)$ with the same frame bounds, for a.e. $t \in \mathbb{T}^n$. If Λ_b^s is an s-eigenvalue of L^*, then, using theorems 3.4 and 7.1, it follows that $\widehat{b}(t)$ is an eigenvalue of $R_s^*(t)$ for a.e. $t \in \sigma_{V_s^b}$. By Theorem 7.6, we conclude that

$$\Big\{P_{J_{V_s^b}}(t)\big(\mathcal{T}_s\phi_i(t)\big) : i \in M, j \in D\Big\}$$

is a frame for $J_s^b(t)$ with bounds $\frac{A}{C(t)}$ and $\frac{B}{C(t)}$ for a.e. $t \in \sigma_{V_s^b}$, where $C(t) = \sum_{j=0}^{d-1} |\widehat{b}(t)|^{2j}$. Using Theorem 3.2, we get

$$1 \leq C(t) = \sum_{j=0}^{d-1} |\widehat{b}(t)|^{2j} \leq \sum_{j=0}^{d-1} \|R_s(t)\|^{2j} \leq \sum_{j=0}^{d-1} \|L\|^{2j}.$$

Therefore, using [7, Theorem 2.3] and Theorem 2.2, the assertion follows. □

Finally, we will prove that the equivalence in Theorem 7.7 holds under additional assumptions. We will say that shift-preserving operator $L : V_s \to V_s$ has the spectral property if there exists $C > 0$ so that $|\lambda_s' - \lambda_s| \geq C$, for all $\lambda_s' \neq \lambda_s$, $\lambda_s, \lambda_s' \in \mathfrak{R}_\lambda(t)$, for a.e. $t \in \sigma_{V_s}$.

Theorem 7.8 ([6], Theorem 3.8). *Let $V_s = S_s(\varphi_1, \ldots, \varphi_d) \subset H^s$ be a shift-invariant space and let $L : V_s \to V_s$ be a normal shift-preserving operator which satisfies the spectral property, with the corresponding range operator R_s. The set*

$$\{L^j \phi_i : \phi_i \in V_s, i \in M, j \in D\}$$

is a frame generator for V_s if and only if

$$\{P_{V_s^b} \phi_i : \phi_i \in V_s, i \in M\}$$

is a frame generator set for V_s^b with the same frame bounds for every s-eigenvalue Λ_b^s of L^*.

Proof. On account of Theorem 7.7, it remains to prove that the opposite implication holds.

Since L is normal operator, by Theorem 7.5, L^* is s-diagonalizable and we can construct an s-diagonalization $(V_s, L^*, b^1, \ldots, b^{\mathbf{q}})$ for L^* so that $\sigma_{V_s^{b^{\ell+1}}} \subseteq \sigma_{V_s^{b^\ell}}$, $\ell = 1, \ldots, \mathbf{q} - 1$, and

$$|\widehat{b^i}(t) - \widehat{b^j}(t)| \geq C, \quad i \neq j, \tag{7.1}$$

for some constant $C > 0$, for a.e. $t \in \mathbb{T}^n$, $i, j = 1, \ldots, \mathbf{q}$ (see Theorem 7.3 and Remark 7.1(3)).

Let $\{P_{V_s^{b^\ell}} \phi_i : i \in M\}$ be a frame generator for $V_s^{b^\ell}$, $\ell = 1, \ldots, \mathbf{q}$, with frame bounds $A, B > 0$. By Theorem 2.2, $\{\mathcal{T}_s(P_{V_s^{b^\ell}} \phi_i)(t) : i \in M\}$ is a frame for $J_s^{b^\ell}(t)$ with the same frame bounds for a.e. $t \in \mathbb{T}^n$, $\ell = 1, \ldots, \mathbf{q}$. Using [7, Theorem 2.3], we get that $\{P_{J_s^{b^\ell}}(t)(\mathcal{T}_s\phi_i(t)) : i \in M\}$ is a frame for $J_s^{b^\ell}(t)$ with the same frame bounds for a.e. $t \in \mathbb{T}^n$, $\ell = 1, \ldots, \mathbf{q}$.

Let $A_\ell := \sigma_{V_s^{b^\ell}} \backslash \sigma_{V_s^{b^{\ell+1}}}$ for $\ell = 1, \ldots, \mathbf{q} - 1$, and $A_{\mathbf{q}} := \sigma_{V_s^{b^{\mathbf{q}}}}$. Then, $\mathbb{T}^n = \bigcup_{\ell=1}^{\mathbf{q}} A_\ell \cup \left(\mathbb{T}^n \backslash \sigma_{V_s}\right)$.

Fix $\widetilde{\ell} \in \{1, \ldots, \mathbf{q}\}$. Then, $\{P_{J_s^{b^\ell}}(t)(\mathcal{T}_s\phi_i(t)) : i \in M\}$ is a frame for $J_s^{b^\ell}$ for a.e. $t \in A_\ell$, $\ell = 1, \ldots, \widetilde{\ell}$. By [6, Theorem 3.5], $\{R_s^j(t)(\mathcal{T}_s\phi_i(t)) : i \in M, j \in$

$D\}$ is a frame for $J_s(t)$ for a.e. $t \in A_{\widetilde{\ell}}$ with bounds

$$A\bigg(\frac{\mathbf{q}}{\beta(t)}\sum_{i=0}^{\widetilde{\ell}-1}\binom{\widetilde{\ell}-1}{i}^2\|R_s(t)\|^{2i}\bigg)^{-1} \quad \text{and} \quad B\bigg(\widetilde{\ell}\sum_{j=0}^{d-1}\|R_s(t)\|^{2j}\bigg),$$

where $\beta(t) = \min_{1\le\ell\le\widetilde{\ell}}\prod_{i=1,i\neq\ell}^{\widetilde{\ell}}|\widehat{b^\ell}(t)-\widehat{b^i}(t)|^2$.

There is no loss of generality in assuming $C<1$ in (7.1). Then, $C^{2\mathbf{q}} \le C^{2\widetilde{\ell}} \le \beta(t)$ for a.e. $t \in A_{\widetilde{\ell}}$, and we get

$$A\bigg(\frac{\mathbf{q}}{C^{2\mathbf{q}}}\sum_{i=0}^{\mathbf{q}-1}\binom{\mathbf{q}-1}{i}^2\|L\|^{2i}\bigg)^{-1} \le A\bigg(\frac{\mathbf{q}}{\beta(t)}\sum_{i=0}^{\widetilde{\ell}-1}\binom{\widetilde{\ell}-1}{i}^2\|R_s(t)\|^{2i}\bigg)^{-1},$$

$$B\bigg(\widetilde{\ell}\sum_{j=0}^{d-1}\|R_s(t)\|^{2j}\bigg) \le B\bigg(\mathbf{q}\sum_{j=0}^{d-1}\|L\|^{2j}\bigg),$$

for a.e. $t \in A_{\widetilde{\ell}}$. For every set $A_{\widetilde{\ell}}$, $\widetilde{\ell}=1,\dots,\mathbf{q}$, we have the same frame bounds and therefore $\{R_s^j(t)(\mathcal{T}_s\phi_i(t)) : i \in M, j \in D\}$ is a frame for $J_s(t)$ for a.e. $t \in \mathbb{T}^n$. Finally, using (3.6) and Theorem 2.2, we conclude that the family $\{L^j\phi_i : i \in M, j \in D\}$ is a frame generator for V_s with the frame bounds $A\big(\frac{\mathbf{q}}{C^{2\mathbf{q}}}\sum_{i=0}^{\mathbf{q}-1}\binom{\mathbf{q}-1}{i}^2\|L\|^{2i}\big)^{-1}$, $B\big(\mathbf{q}\sum_{j=0}^{d-1}\|L\|^{2j}\big)$. □

Acknowledgment

This research was supported by the Science Fund of the Republic of Serbia, #GRANT No 2727,*Global and local analysis of operators and distributions* - GOALS. The authors are also supported by Serbian Ministry of Science and Technology (grant 451-03-47/2023-01/200122), and project F10 of the Serbian Academy of Sciences and Arts.

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Aleksandar Aksentijević
Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia
e-mail: aksentijevic@kg.ac.rs

Suzana Aleksić
Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia
e-mail: suzana.aleksic@pmf.kg.ac.rs

Stevan Pilipović
Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Trg Dositeja Obradovica 4, 21000 Novi Sad, Serbia
e-mail: stevan.pilipovic@dmi.uns.ac.rs