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Critical total strain-based Phase-Field Damage Model for ductile fatigue

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Abstract The structural integrity of engineering components is crucial in engineering design, because they are prone to damage due to repeated loading and unloading, particularly under low-cycle fatigue (LCF) conditions. This study presents a critical strain-based phase-field damage model (PFDM) for ductile fatigue. The model captures the accumulation of damage by relating the critical strain energy to fracture energy release rate. The PFDM formulation is integrated into Finite Element Method (FEM) software, enabling the simulation of damage phenomena under cyclic loading. The model considers the evolution of damage driven by energy-based criteria. A numerical example demonstrates the capability of the approach to predict fatigue-induced failure and validates the model's functionality. This work underscores the importance of energy-based damage indicators for modeling fatigue in ductile materials and a possibility to use critical total strain energy as a material parameter.

1 Introduction

Engineering structures are recognized as components with remarkable strength, ductility, and reliability. However, such structures are susceptible to damage accumulation under cyclic loading. Simulation of low-cycle fatigue (LCF), characterized by significant plastic strain and a limited number of loading cycles, is demanding task in various engineering fields such as aerospace, automotive, and energy.

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Traditional approach which considers S-N curves, cannot capture the mechanisms of damage evolution, especially in ductile materials. There are limitations to model crack initiation and propagation under LCF conditions. On the other side, energy-based approaches provide a more comprehensive framework for understanding the damage process.

Phase-field damage models (PFDM) have gained attention in the last period for their ability to simulate fatigue in materials. These models introduce a continuous damage field variable to represent the state from an intact to a fully damaged [1]. In combination with the finite element method (FEM), PFDM represents a robust method for capturing behavior under cyclic loading conditions [2].

This paper proposes a critical total strain-based PFDM to introduce a critical total strain as a new material parameter. The critical total strain represents the value of strain when the stress starts to decrease for uniaxial loading conditions. By relating the critical total strain to fracture energy release rate, the proposed model introduces alternative material parameters that better reflect ductile behavior. A novel implementation into the in-house Finite Element Method (FEM) software, PAK-DAM v24 is analyzed in details. The formulation takes into account the elastic, plastic, and fracture contributions to the total strain energy, making it suitable for simulating ductile fatigue under cyclic loading. The implementation within FEM software enables accurate simulations of fatigue-induced damage in metallic structures under LCF conditions. Numerical example validates the model's effectiveness, demonstrating its ability to follow the damage accumulation phenomenon.

2 Critical total strain based PFDM

In ductile materials, a significant plastic strain occurs before fracture, which need a non-linear stress-strain relationship beyond the yield stress.

The PFDM method has emerged as a powerful approach for simulating damage in materials. Unlike classical fracture mechanics approaches, PFDM uses a continuous damage variable, $d \in [0, 1]$, where $d = 0$ represents the undamaged state and $d = 1$ denotes total material damage.

A background for the PFDM is a Griffith's theory [3]. The equilibrium of the surface and the bulk strain energy defines a criterion for damage in the material. Francfort and Marigo [4] proposed regularized variational approach of minimization of an energy functional [5]. A damage field variable, d is a scalar measure of cracks in a material. The regularized crack functional for the multi-dimensional problems is [6]:

$$\Gamma_l (d) = \int_V \gamma_l (d, \nabla d) \, dV \tag{1}$$

where the crack surface density function per unit volume is [6, 7]:

$$\gamma_l = \frac{1}{2l_c} \left(d^2 + l_c^2 |\nabla d|^2 \right) \tag{2}$$

and ∇ is the gradient operator and l_c is the characteristic width of a crack.

2.1 Strain Energy Equilibrium in a Damaged Material

A total strain ε can be decomposed as $\varepsilon = \varepsilon_e + \varepsilon_p$, where ε_e is an elastic strain, and ε_p is a plastic strain. Internal potential energy density P_{int} consists of an elastic energy P_e, a plastic energy P_p, and a fracture term P_f as [7]:

$$P_{\text{int}} = P_e + P_p + P_f. \tag{3}$$

The elastic and plastic components are derived from material constitutive laws, while the fracture energy density is governed by the phase-field framework. The elastic bulk strain energy is [8]

$$P_e = (1-d)^2 \bar{\omega}_e \tag{4}$$

where $\bar{\omega}_e$ represents an effective elastic strain energy density per unit volume:

$$\bar{\omega}_e = \frac{1}{2}\sigma_0 : \varepsilon_e \tag{5}$$

and σ_0 is an effective Cauchy stress tensor. The total plastic strain energy density is [7]:

$$P_p = (1-d)^2 \bar{\omega}_p. \tag{6}$$

An effective plastic strain energy density per unit volume $\bar{\omega}_p$ is [1, 9, 10]:

$$\bar{\omega}_p = \sigma_{yv}\bar{\varepsilon}_p + \left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(\bar{\varepsilon}_p + \frac{1}{n}e^{-n\bar{\varepsilon}_p}\right) + \frac{1}{2}H\bar{\varepsilon}_p^2 \tag{7}$$

where σ_{yv} is a yield stress, $\sigma_{y0,\infty}$ is a saturation hardening stress, while n and H are a hardening exponent and modulus.

In ductile materials, the damage does not increase immediately after the loading is applied. The "work-densities-based criterion with threshold" is defined by Miehe et al. in [7] through the fracture surface energy density as:

$$P_f = G_v\left[d + \frac{l_c^2}{2}|\nabla d|^2\right] \tag{8}$$

where $G_v = G_c/l_c$ is a specific fracture energy per unit volume [10]. The damage d is in linear relation in eq. (8) [7]. The eq. (8) is further derived in relation to γ_l in eq. (2), as follows [8]:

$$P_f = G_v l_c \gamma_l + \frac{G_v}{2} - (1-d)^2\frac{G_v}{2} \tag{9}$$

A critical total strain energy density value [8] is

$$\bar{\omega}_{cr} = \frac{G_v}{2}. \tag{10}$$

The total internal potential energy density is:

$$P_{\text{int}} = (1-d)^2 \left(\bar{\omega}_e + \bar{\omega}_p - \bar{\omega}_{cr}\right) + \bar{\omega}_{cr} + G_v l_c \gamma_l. \tag{11}$$

The total internal energy potential can be calculated as:

$$\Psi = \int_V P_{\text{int}} dV = \int_V \left\{(1-d)^2 \left(\bar{\omega}_e + \bar{\omega}_p - \bar{\omega}_{cr}\right) + \bar{\omega}_{cr} + G_v l_c \gamma_l\right\} dV \tag{12}$$

The variation of the expanded form of total internal energy potential [8]:

$$\begin{aligned}\delta\Psi = \int_V &\left\{\sigma : \delta\varepsilon_e + \frac{1}{2} g'(d)\, \varepsilon_e^T : \sigma_0 \delta d + g'(d)\left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(\bar{\varepsilon}_p + \frac{1}{n} e^{-n\bar{\varepsilon}_p}\right)\delta d + \right.\\
&+ g'(d)\,\frac{1}{2} H \bar{\varepsilon}_p^2 \delta d + g'(d)\,\sigma_{yv}\bar{\varepsilon}_p \delta d - g'(d)\,\frac{G_v}{2} + G_v\left[d\delta d + l_c^2 \nabla d \nabla \delta d\right] + \\
&\left. + \left(-g(d)\,\sigma_0 : \frac{\partial \varepsilon_p}{\partial \bar{\varepsilon}_p} + g(d)\left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(1 - e^{-n\bar{\varepsilon}_p}\right) + g(d)\, H\bar{\varepsilon}_p + g(d)\,\sigma_{yv}\right)\delta\bar{\varepsilon}_p\right\} dV\end{aligned} \tag{13}$$

where $g(d)$ is a degradation function and its derivative $g'(d)$ over d [11, 12]:

$$g(d) = (1-d)^2 \tag{14}$$

$$g'(d) = -2(1-d) \tag{15}$$

Equilibrium of internal and external potential energy can be expressed as [8]:

$$\begin{aligned}\int_v &\left\{\sigma : \delta\varepsilon_e + \frac{1}{2} g'(d)\, \varepsilon_e^T : \sigma_0 \delta d + g'(d)\left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(\bar{\varepsilon}_p + \frac{1}{n} e^{-n\bar{\varepsilon}_p}\right)\delta d + \right.\\
&+ g'(d)\,\frac{1}{2} H \bar{\varepsilon}_p^2 \delta d + g'(d)\,\sigma_{yv}\bar{\varepsilon}_p \delta d - g'(d)\,\frac{G_v}{2} + G_v\left[d\delta d + l_c^2 \nabla d \cdot \nabla \delta d\right] + \\
&\left. + \left(-g(d)\,\sigma_0 : \frac{\partial \varepsilon_p}{\partial \bar{\varepsilon}_p} + g(d)\left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(1 - e^{-n\bar{\varepsilon}_p}\right) + g(d)\, H\bar{\varepsilon}_p + g(d)\,\sigma_{yv}\right)\delta\bar{\varepsilon}_p\right\} dV \\
&= \int_V \mathbf{b} \cdot \delta\mathbf{u} dV + \int_A \mathbf{h} \cdot \delta\mathbf{u} dA\end{aligned} \tag{16}$$

where $\mathbf{b}$ is a body force field per unit volume, and $\mathbf{h}$ is a boundary traction per unit area, $\mathbf{u}$ is a displacement vector. By applying the Gauss theorem and total derivatives [9] and the Neumann-type boundary conditions the equilibrium equation is [8, 13]:

$$Div\left[\sigma\right] + \mathbf{b} = 0 \tag{17}$$

the plasticity yield condition law [9]:

$$\bar{\sigma}_{eq} - \sigma_{yv} - \left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(1 - e^{-n\bar{\varepsilon}_p}\right) - H\bar{\varepsilon}_p = 0 \tag{18}$$

and the phase-field damage evolution law [7]:

$$G_v\left[d - l_c^2\nabla^2 d\right] + g'(d)\max\left(\bar{\omega}_e + \bar{\omega}_p - \frac{G_v}{2}\right) = 0 \tag{19}$$

The fatigue function $f(\bar{\alpha}(t))$, dependent on the loading history, introduces time-dependence by accounting for the accumulation of damage under cyclic loading. The phase-field damage evolution law described by eq. (19) can be extended to take into account accumulation of damage as follows [8]:

$$f(\bar{\alpha}(t))\,G_v\left[d - l_c^2\nabla^2 d\right] + g'(d)\max\left(\bar{\omega}_e + \bar{\omega}_p - \frac{G_v}{2}\right) = 0. \tag{20}$$

The cyclic loading produces a damage, which is accumulated by a fatigue function $f(\bar{\alpha})$ defined to capture that phenomenon [15, 16, 17]. The increment of the history variable can be defined as [15, 16, 17, 18]:

$$\Delta\bar{\alpha}(t) = \int_t^{t+\Delta t} H(\alpha\dot{\alpha})\,|\dot{\alpha}|\,d\tau, \tag{21}$$

where $H(\alpha\dot{\alpha})$ is the Heaviside step function [15, 16, 17]. The fatigue history variable α is defined as [15, 16, 17].:

$$\alpha = P_{int}, \tag{22}$$

what gives:

$$^{t+\Delta t}\bar{\alpha} = {}^{t}\bar{\alpha} + \Delta\bar{\alpha}, \tag{23}$$

while the fatigue function is [16]:

$$f(\bar{\alpha}) = \begin{cases} 1 & if\ \bar{\alpha} \le \omega_{cr} \\ \left(\frac{2\alpha_T}{\bar{\alpha}+\alpha_T}\right)^2 & \text{if } \bar{\alpha} > \omega_{cr} \end{cases} \tag{24}$$

Here, ω_{cr} represents a threshold value, below which the fracture energy is unaffected.

2.2 Critical fracture strain energy density

The main conclusion from Equation (10) is that the critical strain energy density per unit volume $\bar{\omega}_{cr}$, is equal to the threshold value of fracture energy $G_v/2$. This is amount of the strain energy density which cannot be consumed on damaging. It is equal to the hatched area below the stress-strain curve in Figure 1, which is based on one-dimensional problem [7]. The area under the linear part of the diagram can be calculated as the area of triangle:

$$A_{linear} = \frac{1}{2}\sigma_{yv}\varepsilon_y = \frac{\sigma_{yv}^2}{E} \tag{25}$$

The are below the non-linear part marked as A in Figure 1, can be calculated as an integral of yielding function eq. (18):

$$A = \int_{\varepsilon_y}^{\varepsilon_{cr}} \left(\sigma_{yv} + \left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left(1 - e^{-n\bar{\varepsilon}_p}\right) + H\bar{\varepsilon}_p\right) d\bar{\varepsilon}_p \tag{26}$$

The total area can be calculated as:

$$\begin{aligned} \bar{\omega}_{cr} = A_{linear} + A = \frac{\sigma_{yv}^2}{2E} + \sigma_{yv}\left(\varepsilon_{cr} - \frac{\sigma_{yv}}{E}\right) + \\ + \left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left[\left(\varepsilon_{cr} - \frac{\sigma_{yv}}{E}\right) + \frac{1}{n}\left(e^{(-n\varepsilon_{cr})} - e^{\left(-n\frac{\sigma_{yv}}{E}\right)}\right)\right] + \\ + \frac{H}{2}\left(\varepsilon_{cr}^2 - \frac{\sigma_{yv}^2}{E^2}\right) \end{aligned} \tag{27}$$

where $\varepsilon_y = \frac{\sigma_{yv}}{E}$ is the yield strain and E represents Young's modulus.

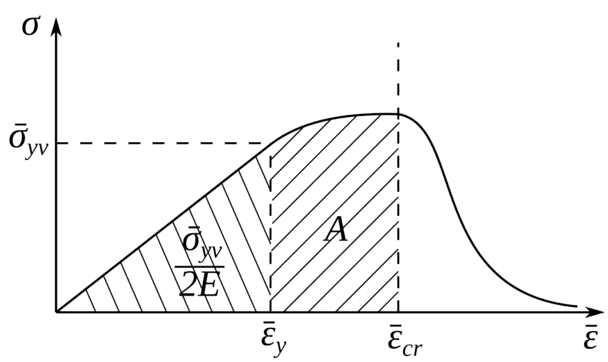

Fig. 1 Relationship between the threshold fracture energy value and the effective critical strain value

From eq. (10), the critical fracture energy per unit volume can be calculated as [7]:

$$\begin{aligned} G_v = \frac{\sigma_{yv}^2}{E} + 2\sigma_{yv}\left(\varepsilon_{cr} - \frac{\sigma_{yv}}{E}\right) + \\ + 2\left(\sigma_{y0,\infty} - \sigma_{yv}\right)\left[\left(\varepsilon_{cr} - \frac{\sigma_{yv}}{E}\right) + \frac{1}{n}\left(e^{(-n\varepsilon_{cr})} - e^{\left(-n\frac{\sigma_{yv}}{E}\right)}\right)\right] + \\ + H\left(\varepsilon_{cr}^2 - \frac{\sigma_{yv}^2}{E^2}\right) \end{aligned} \tag{28}$$

Table 1 Material parameters

E[MPa]	ν[-]	σ_{yv}[MPa]	$\sigma_{y0,\infty}$[MPa]	H [MPa]	n[-]	l_{cr}[mm]	$\bar{\varepsilon}_{cr}$-[]
199000	0.29	345	635	9	18	0.1	0.012

Consequently, a critical value of total strain $\bar{\varepsilon}_{cr}$ can be defined from the stress–strain diagram obtained in the experiments; and with the known Young's modulus and the yield stress, the critical fracture energy release rate per unit volume can be determined as internal material parameter by eq. (28).

3 Numerical test

The critical total strain-based PFDM implementation response is presented by a simulation of the unit cube example, which can be considered as a segment of a larger structure. Symmetry boundary conditions are defined at the bottom and the two orthogonal sides of the cube (Figure 3). The upper side of the cube is loaded by unit prescribed displacements, multiplied by the function shown in Figure 2. The cyclic loading conditions are achieved to evaluate the model's predictive capabilities. The unit dimensions of the cube model are chosen to exclude the scale and the mesh sensitivity of the structure and to show the functionality of the proposed implementation. The material parameters (Table 1) are proposed as generic parameters of a ductile material.

As it can be observed at Figure 4, after the initial linear stress-strain relationship, up to the initial yield stress of 345MPa, the plastic strain occurs and the hardening of the material according to the plasticity yield condition (eq. 18) can be noticed. When the total strain achieves the critical total strain value of $\bar{\varepsilon}_{cr} = 0.012$, the threshold value of $G_v/2$ is achieved and the damage in the material occurs, while the stress value decreases. By further loading, at the 2% of total strain, the material is unloaded and than again the loading is repeated up to 3%. The same loading-unloading is repeated up to 4%. The decrease of material stiffness at each loading-unloading cycle is visible by decreasing of linear stress-strain dependence (Young modulus).

4 Conclusion

The proposed PFDM based on critical total strain effectively captures the interplay between elastic, plastic, and fracture contributions in ductile materials under cyclic loading. It provides a robust framework for predicting fatigue-induced damage, making the model a valuable tool for simulating low-cycle fatigue in engineering applications. The relationship between the critical total strain and the fracture energy release rate provide a possibility for more straightforward material parameters iden-

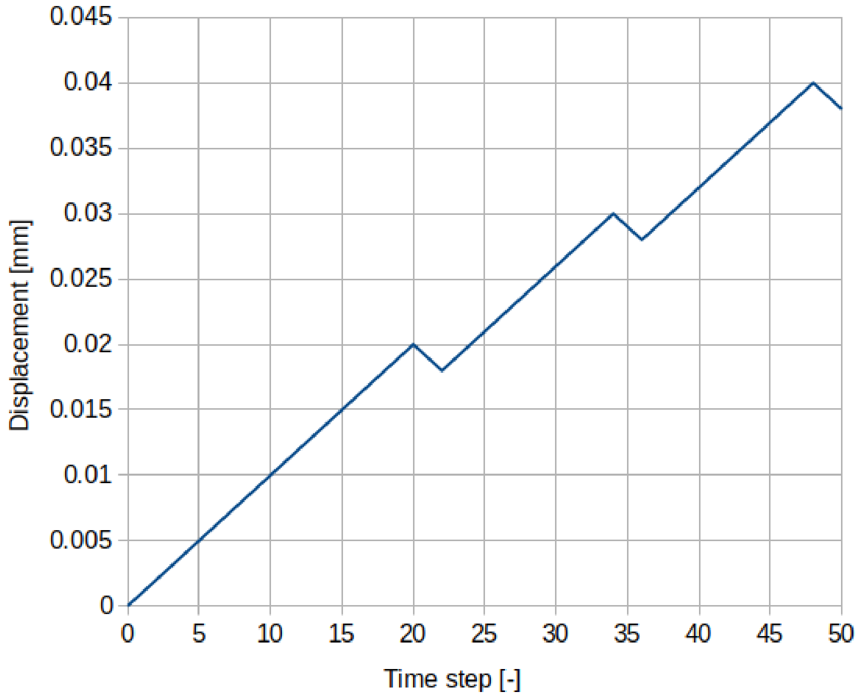

Fig. 2 Displacement control over time steps period.

tification. Further testing of this PFDM implementation by more complex boundary and loading conditions will be done as future research task.

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Competing Interests The authors have no conflicts of interest to declare that are relevant to the content of this chapter.

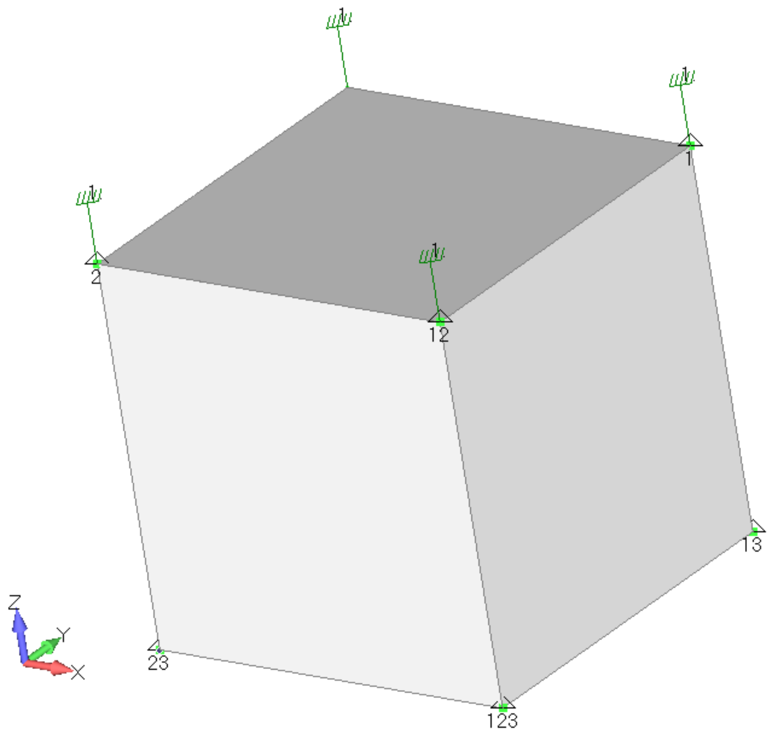

Fig. 3 Boundary and loading conditions.

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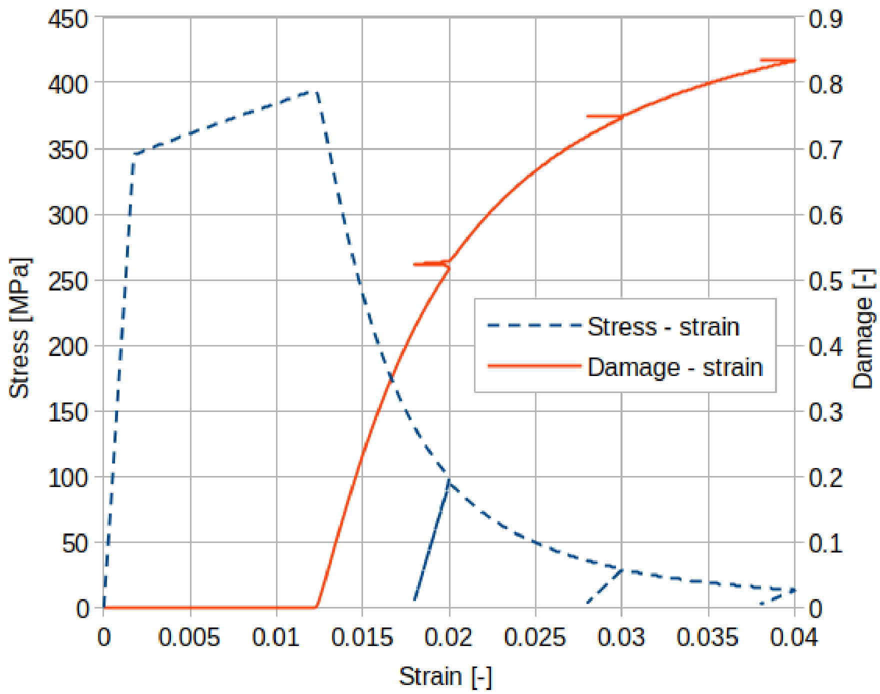

Fig. 4 Stress-strain and damage-strain response for cyclic loading.

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