

## SESSION A:

## EARTH MOVING AND TRANSPORTATION MACHINERY

## The determination of the deflection of the stepped column by the finite difference method

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### ABSTRACT

Besides the strength criterion, steel structures must also meet the deformation criterion, which in some cases may be critical. Excessive deformation of the structure may lead to additional loads and improper operation of the installed equipment. The most common practice in steel structure design involves using the finite element method (FEM) to calculate the stress and deformation of the current model. However, this is usually limited to the case study for the particular structural model. The paper demonstrates the application of the finite difference method (FDM) to determine the deflection of a stepped column, enabling a broader parametric study and analysis of a variety of solutions that satisfy the deformation criterion. The application of FDM yields a scheme convenient for coding in MATLAB while showing the dependence between the column deflection and the design parameters. The results from the finite element analysis (FEA) are in excellent agreement with those from the FDM.

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### KEYWORDS

Stepped Column, Deflection, Finite Difference Method, Finite Element Method.

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### 1. INTRODUCTION

The steel structures must meet many requirements, such as strength, deformation, elastic and dynamic stability, and connection safety. It is often the case that the whole structure and/or its parts have long lengths, so the maximum deformation criterion may be the most restrictive for its design. This criterion is defined by permissible deformation at characteristic points of the structure under maximum operational load. Within exploitation, the steel structure's lack of stiffness can lead to excessive deformation and the development of additional static and dynamic loads. Further on, this may eventually jeopardize operational safety and shorten the operational life of the subassemblies or the entire facility. So, the limitation of deformation provides good exploitation features and a long equipment lifespan.

For instance, for each crane type, there are prescribed maximum deflection values for characteristic points on its carrying structure. Permissible static deflection at the mid-span point of the bridge crane is  $L/600$  [1], where  $L$  denotes the span. For the case of the gantry cranes, there is a prescription for the static deflection values for the mid-span point and the endpoint of the overhang: it is  $(1/1000 \div 1/600)L$  for the mid-span, while it is  $(1/400 \div 1/200)L$  for the overhang endpoint, where  $L$  is the distance between the gantry crane legs, and  $L_1$  denotes the overhang length [2].

If the steel supporting structure has an unchangeable geometry and consists of beams with uniform cross-sections, it is relatively easy to calculate its deflection. Nevertheless, it is a rather rare situation. The typical example of a structure with a variable-geometry configuration that changes with the working position is an articulated boom in a truck crane [3]. However, if the cross-sections of the segments are uniform, it is possible, with some simplifications, to develop an analytical model for calculating the boom tip displacements [4]. Building an analytical model to calculate displacements of characteristic points enables the use of deformation conditions to optimize the supporting structure and reduce its mass [5-9].

In more complex structures, deflection checks are typically conducted using FEM models [10-12]. It is usually done for the final design version. This occurs in the late design phase, when it is far more difficult, and sometimes even impossible,

to implement design changes to the structural model if it does not meet the displacement criteria for characteristic points. Hence, there is a need for an approach applicable in the early design phase to provide a broader perspective on possible design solutions that meet the permissible deformation criteria.

The predominant method for the structural analysis of steel structures is FEM. Nevertheless, the authors employ additional numerical methods to compute deflection. For example, the angle of rotation is represented by a power series and substituted into the derived governing nonlinear differential equation for bending of prismatic and non-prismatic inextensible beams [13]. Others used the FDM to develop the numerical scheme for solving the deformation problem of beams in bending [14-16].

The paper presents the FDM application procedure for determining the deflection of the supporting structure in a simple example of a multi-segment column subjected to a force at the top. This approach is also applicable to any load combination and number of segments.

## 2. THE MATHEMATICAL MODEL

The model used to present the proposed FDM approach for deflection determination is shown in Fig. 1. To make things as simple as possible, it is a three-segment column made of steel pipes with different diameters  $D_1$ ,  $D_2$ , and  $D_3$ . Nevertheless, this approach is applicable to any type of cross-section. To make the expressions simpler, but with no derogation of the generality and applicability of the method, the wall thicknesses of all pipes are taken to be equal ( $t$ ). The column is fixed at point  $A$ , while points  $B$  and  $C$  are the joints between the segments.

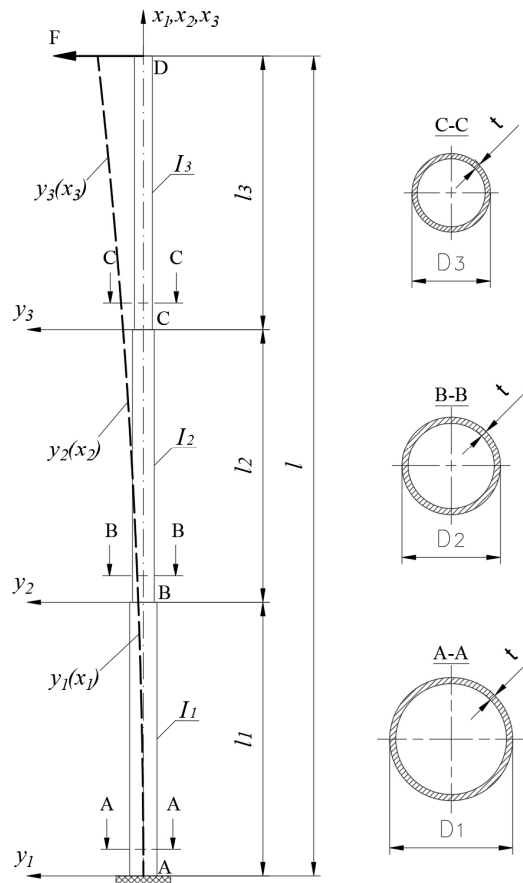


Figure 1: A column with stepped changes of cross-section

The lengths of the segments are  $l_1$ ,  $l_2$  and  $l_3$  and the column total length is  $l=l_1+l_2+l_3$ . The point force  $F$ , acting transversely at the top, is bending the column and the deformed shape is shown as dashed elastic line. Each segment has its own coordinate system  $x_j y_j$  and its own deflection function  $y_j=y_j(x_j)$ ,  $j=1,2,3$ . If the following ratios of diameters are introduced

$$p = \frac{D_2}{D_1}, \quad q = \frac{D_3}{D_2} \quad (1)$$

the moments of inertia for the  $z_i$  axes of the segments are respectively as follows:

$$I_1 = \frac{\pi [D_1^4 - (D_1 - 2t)^4]}{64} \quad (2)$$

$$I_2 = \frac{\pi [D_2^4 - (D_2 - 2t)^4]}{64} = \frac{\pi D_1^4}{64} \left[ p^4 - \left( p - 2 \frac{t}{D_1} \right)^4 \right] \quad (3)$$

$$I_3 = \frac{\pi [D_3^4 - (D_3 - 2t)^4]}{64} = \frac{\pi D_2^4}{64} \left[ q^4 - \left( q - 2 \frac{t}{D_2} \right)^4 \right] \quad (4)$$

The bending moment at the distance  $x$  from the column support is

$$M = F(l - x) \quad (5)$$

The governing bending equations for each segment are as follows:

$$\frac{d^2 y_j(x_j)}{dx_j^2} = \frac{M}{B_j}, \quad j = 1, 2, 3 \quad (6)$$

The boundary conditions define the values of displacements and the slopes at points A, B and C. For point A they are:

$$y_1(x_1 = 0) = 0 \quad (7)$$

$$\frac{dy_1}{dx_1}(x_1 = 0) = 0 \quad (8)$$

The boundary conditions for point B are:

$$y_1(x_1 = l_1) = y_2(x_2 = 0) \quad (9)$$

$$\frac{dy_1}{dx_1}(x_1 = l_1) = \frac{dy_2}{dx_2}(x_2 = 0) \quad (10)$$

The boundary conditions for point C are:

$$y_2(x_2 = l_1 + l_2) = y_3(x_3 = 0) \quad (11)$$

$$\frac{dy_2}{dx_2}(x_2 = l_1 + l_2) = \frac{dy_3}{dx_3}(x_3 = 0) \quad (12)$$

### 3. THE FDM DISCRETIZATION

Fig. 2 shows FDM grid scheme for the numerical solution of the deflection line. The length of the beam  $l$  is equally divided by  $N=l/s$  grid points (nodes) into  $N$  segments with length  $s = L / N$ . Later transformation of the boundary conditions includes the support point A denoted as node 0 and one fictitious node on the left side of the support, denoted as node  $-l$ . As the displacements of fictitious nodes are designated by superscripted  $f$ , the displacement of this fictitious node is denoted as  $Y_{-l}^f$ .

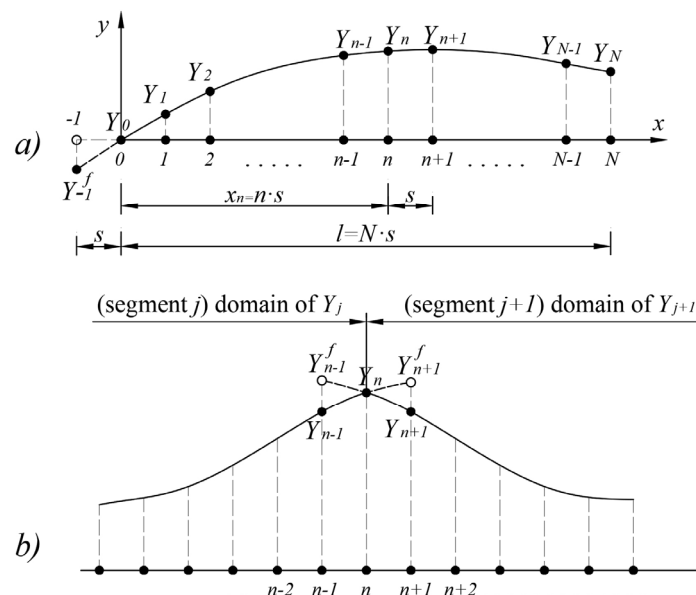


Figure 2: a) FDM grid scheme; b) fictitious nodes at adjacent segments' junction

According to the central FDM, approximations for derivatives of the deflection function  $y(x)$  in node  $n$  are as follows:

$$\left(\frac{dy}{dx}\right)_n \approx \frac{-Y_{n-1} + Y_{n+1}}{2s} \quad (13)$$

$$\left(\frac{d^2y}{dx^2}\right)_n \approx \frac{Y_{n-1} - 2Y_n + Y_{n+1}}{s^2} \quad (14)$$

After the introduction of bending stiffnesses for the segments as  $B_j = EI_j$ ,  $j=1,2,3$  and discretization of the global longitudinal coordinate to  $x=ns$ , governing bending equations (6) can be transformed to the following algebraic formulation:

$$Y_{n-1} - 2Y_n + Y_{n+1} = \frac{s^2 F}{B_j} (l - ns), \quad j=1,2,3 \quad (15)$$

An FDM transformation of the boundary conditions in support A yields:

$$Y_0 = 0 \quad (16)$$

$$\frac{-Y_{-1}^f + Y_1}{2s} = 0 \Rightarrow Y_{-1}^f = Y_1 \quad (17)$$

Geometrical interpretation of the discretized boundary conditions (16) and (17) is shown in Fig. 3.

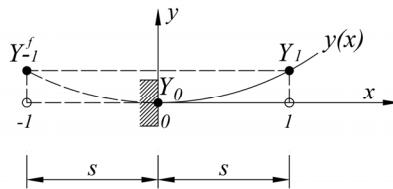


Figure 3: Deflection function around the fixed support

If the node in point B is designated as  $b=l_1/s$ , the algebraic boundary conditions are as follows:

$$Y_b = Y_b^f \quad (18)$$

$$-Y_{b-1} + Y_{b+1}^f = -Y_{b-1}^f + Y_{b+1} \quad (19)$$

Since displacements of all fictitious nodes must be eliminated from the discretized governing equations (15), an additional condition related to equality of bending moments is introduced:

$$B_1 \frac{d^2 y_1}{dx_1^2}(x_1 = l_1) = B_2 \frac{d^2 y_2}{dx_2^2}(x_2 = 0) \Rightarrow B_1 \frac{Y_{b-1} - 2Y_b + Y_{b+1}^f}{s^2} = B_2 \frac{Y_{b-1}^f - 2Y_b^f + Y_{b+1}}{s^2} \quad (20)$$

Combining (18), (19) and (20) yields:

$$Y_{b+1}^f = (1 - 2P_1)Y_{b-1} + (2P_1 - 2P_2)Y_b + (1 - P_1 + P_2)Y_{b+1} \quad (21)$$

where  $P_1 = \frac{B_1}{B_1 + B_2}$ ,  $P_2 = \frac{B_2}{B_1 + B_2}$ .

Similarly to the previous, the algebraic boundary conditions for the node at point C are derived. If the node in point C is designated as  $c=l_1+l_2/s$ , the algebraic boundary conditions are:

$$Y_c = Y_c^f \quad (22)$$

$$-Y_{c-1} + Y_{c+1}^f = -Y_{c-1}^f + Y_{c+1} \quad (23)$$

Since displacements of all fictitious nodes must be eliminated from the discretized governing equations (15), an additional condition related to equality of bending moments is introduced:

$$B_2 \frac{d^2 y_2}{dx_2^2}(x_2 = l_2) = B_3 \frac{d^2 y_3}{dx_3^2}(x_3 = 0) \Rightarrow B_2 \frac{Y_{c-1} - 2Y_c + Y_{c+1}^f}{s^2} = B_3 \frac{Y_{c-1}^f - 2Y_c^f + Y_{c+1}}{s^2} \quad (24)$$

Combining (22), (23) and (24) yields:

$$Y_{c+1}^f = (1 - 2P_3)Y_{c-1} + (2P_3 - 2P_4)Y_c + (1 - P_3 + P_4)Y_{c+1} \quad (25)$$

where  $P_3 = \frac{B_2}{B_2 + B_3}$ ,  $P_4 = \frac{B_3}{B_2 + B_3}$ .

Now, a set of  $N$  algebraic equations can be written for the grid nodes  $n=0, 1, \dots, N-1$ , with the node displacements  $Y_i$ ,  $i=1, \dots, N$  as unknowns. Thereby, the equations of the deflection function written for the nodes  $0$  and  $1$  include displacements  $Y_0$  and  $Y_{-1}$ , which are eliminated by using transformed boundary conditions (16) and (17). Also, the fictitious displacements in algebraic bending equations written for nodes  $b$  and  $c$  are eliminated using (18), (21), (22) and (25). Finally, a system of linear algebraic equations, very convenient for programming (e.g. in MATLAB), has the form:

$$\begin{aligned}
 n=0 &\rightarrow 2Y_1 = \frac{s^2 Fl}{B_1} \\
 n=1 &\rightarrow -2Y_1 + Y_2 = \frac{s^2 F}{B_1}(l-s) \\
 n=2 \div (b-1) &\rightarrow Y_{n-1} - 2Y_n + Y_{n+1} = \frac{s^2 F}{B_1}(l-ns) \\
 n=b &\rightarrow (2-2P_1)Y_{b-1} + (2P_1-2P_2-2)Y_b + (1-P_1+P_2)Y_{b+1} = \frac{s^2 F}{B_1}(l-bs) \\
 n=b+1 &\rightarrow Y_b - 2Y_{b+1} + Y_{b+2} = \frac{s^2 F}{B_2}[l-(b+1)s] \\
 n=(b+2) \div (c-1) &\rightarrow Y_{n-1} - 2Y_n + Y_{n+1} = \frac{s^2 F}{B_2}(l-ns) \\
 n=c &\rightarrow (2-2P_3)Y_{c-1} + (2P_3-2P_4-2)Y_c + (1-P_3+P_4)Y_{c+1} = \frac{s^2 F}{B_2}(l-cs) \\
 n=c+1 &\rightarrow Y_c - 2Y_{c+1} + Y_{c+2} = \frac{s^2 F}{B_3}[l-(c+1)s] \\
 n=(c+2) \div (N-1) &\rightarrow Y_{n-1} - 2Y_n + Y_{n+1} = \frac{s^2 F}{B_3}(l-ns)
 \end{aligned} \tag{26}$$

#### 4. NUMERICAL EXAMPLE AND COMPARISON WITH FEM MODEL RESULTS

The derived system of  $N$  algebraic equations yields the column node displacements  $Y_i$ ,  $i=1, \dots, N$ . The numerical example has the following set of input parameters:  $t=3 \text{ mm}$ ,  $l_1=l_2=l_3=1000 \text{ mm}$ ,  $D_1=100 \text{ mm}$ ,  $F=100 \text{ N}$ . The spacing for creating the node grid is set to  $s=10 \text{ mm}$ . The values of the non-dimensional ratios between the pipes' diameters  $p$  and  $q$  are varied between 0.6 and 0.9. The modulus of elasticity is taken as  $E=2.1 \cdot 10^5 \text{ N/mm}^2$ . The solution of the system (26) is obtained using MATLAB, and the FEM analyses are performed in SAP. Table 1 presents the comparison of the results obtained from both (FDM and FEM) numerical approaches for the deflection at the top of the column  $Y_N$ . The relative deviations between the results ( $\delta$ ) are up to 0.2%.

Table 1: Comparison of the results obtained in FDM and FEM approach

| $p$ | $q$ | $Y_N [\text{mm}]$ |          | $ \delta  [\%]$ |
|-----|-----|-------------------|----------|-----------------|
|     |     | FDM               | FEM      |                 |
| 0.6 | 0.6 | 11.5996           | 11.61457 | 0.13%           |
|     | 0.7 | 10.1390           | 10.15290 | 0.14%           |
|     | 0.8 | 9.3530            | 9.36604  | 0.14%           |
|     | 0.9 | 8.8932            | 8.90554  | 0.14%           |
| 0.7 | 0.6 | 8.1899            | 8.20309  | 0.16%           |
|     | 0.7 | 7.3102            | 7.32244  | 0.17%           |
|     | 0.8 | 6.8340            | 6.84545  | 0.17%           |
|     | 0.9 | 6.5540            | 6.56498  | 0.17%           |
| 0.8 | 0.6 | 6.3371            | 6.34892  | 0.19%           |
|     | 0.7 | 5.7671            | 5.77814  | 0.19%           |
|     | 0.8 | 5.4572            | 5.46754  | 0.19%           |
|     | 0.9 | 5.2744            | 5.28424  | 0.19%           |

| $p$ | $q$ | $Y_N [mm]$ |         | $ \delta  [\%]$ |
|-----|-----|------------|---------|-----------------|
|     |     | FDM        | FEM     |                 |
| 0.9 | 0.6 | 5.2450     | 5.25583 | 0.21%           |
|     | 0.7 | 4.8550     | 4.86507 | 0.21%           |
|     | 0.8 | 4.6422     | 4.65166 | 0.20%           |
|     | 0.9 | 4.5163     | 4.52535 | 0.20%           |

Fig. 4 shows the 3D diagram of the FDM results, while Fig. 5 graphically shows the comparison of FDM and FEM results.

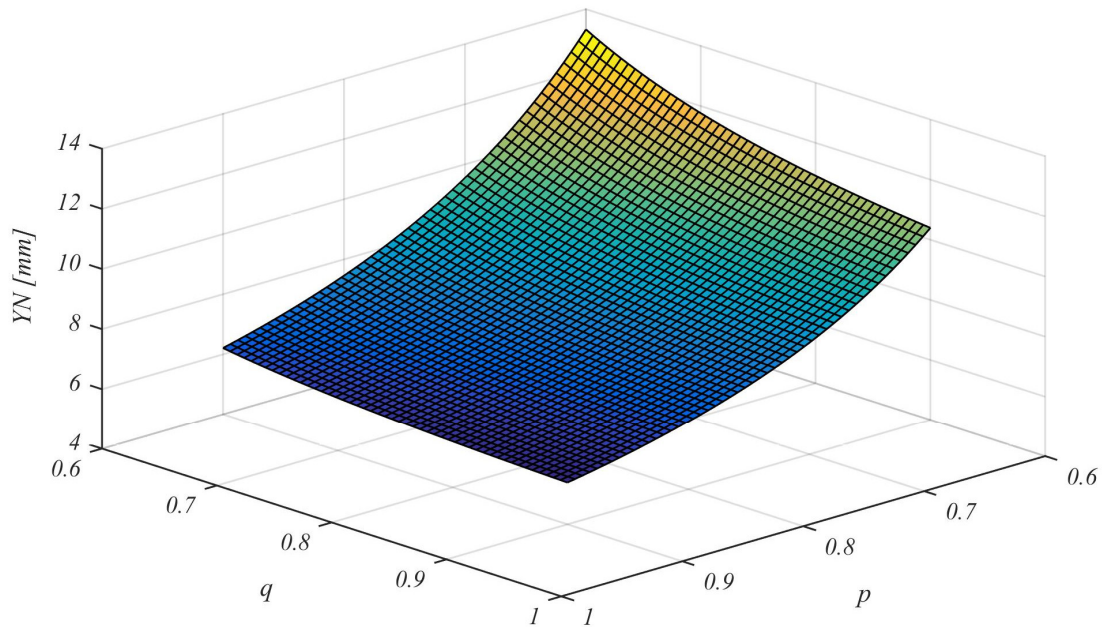


Figure 4: 3D diagram of the results

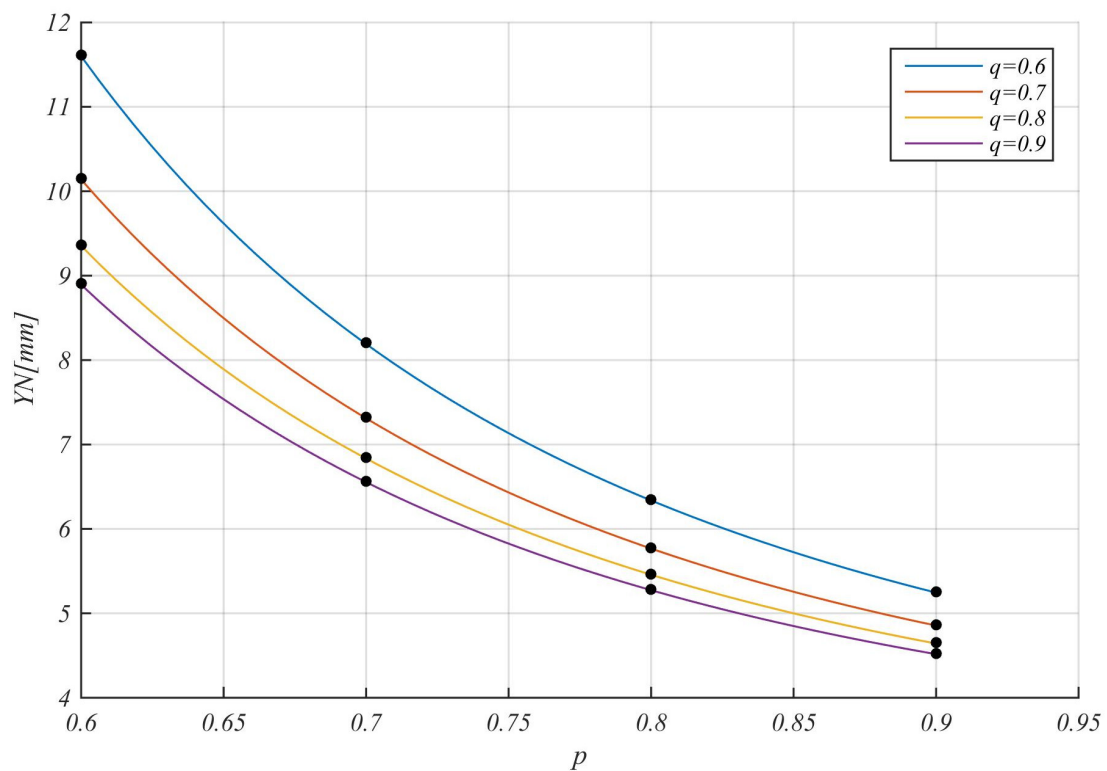


Figure 5: The comparison of the FDM (lines) and FEM (dots) results

## 5. CONCLUSION

As expected, the higher relative coefficients  $p$  and  $q$  the stiffer the column, and the smaller the deflection. The FDM approach for calculating the column deflection yielded results practically identical to those of the FEM approach, with relative deviations up to 0.20%. The FDM model is very convenient for coding mathematical calculations in software, e.g., MATLAB. Once established, the FDM mathematical model enables broader analysis of the structure by varying the geometrical parameters or their ratios. It is possible to track the displacement of any node in response to changes in selected design parameters, e.g., the lengths of the segments.

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