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Analytical Benchmark Verification of Central Difference Time Integration and Explicit FEM Models for the One-Dimensional Wave Equation

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Abstract

This paper presents an analytical and numerical benchmark verification framework for explicit time integration procedures applied to the one-dimensional wave equation. The study combines an exact analytical solution, a closed-form discrete central-difference solution, and explicit finite element models implemented in LS-DYNA and in the in-house academic FEM code PAK-Multiphysics. Two benchmark problems with the same first sinusoidal spatial mode and homogeneous Dirichlet boundary conditions are considered. The first problem, defined by sinusoidal initial displacement and zero initial velocity, is used to analyse the central-difference discretization and its convergence behaviour. The closed-form discrete response enables direct comparison with the analytical solution and confirms the expected second-order accuracy under coupled mesh and time-step refinement at a fixed Courant number. The second problem, defined by zero initial displacement and sinusoidal initial velocity, introduces a phase-shifted temporal response suitable for explicit finite element verification. Four meshes are analysed over five periods of the first longitudinal mode. The numerical responses are assessed using displacement histories, maximum absolute errors, RMS errors, relative RMS errors, amplitude errors, and an energy check. The PAK-Multiphysics results show very close agreement with the analytical solution and a systematic reduction of the error measures, consistent with the lumped-mass central-difference formulation. The LS-DYNA results provide an independent commercial-code comparison, showing decreasing displacement errors under refinement and bounded total-energy variation. The proposed framework provides a transparent and reproducible benchmark for verifying one-dimensional explicit wave propagation models.



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1. Introduction

Explicit time integration methods are widely used in structural dynamics, impact mechanics, wave propagation problems, and high-strain-rate simulations, where the response

of the system is governed by rapidly propagating disturbances. In this class of problems, the central difference method represents one of the most important numerical procedures because of its simple algorithmic structure, conditionally stable character, and direct suitability for explicit finite element formulations [1–4]. Classical time integration procedures form the basis for transient structural dynamic analysis [1,4,5]. The Newmark method represents one of the fundamental developments in computational structural dynamics [6], while the explicit central difference scheme remains particularly suitable for wave propagation, impact, and high-speed transient problems because it avoids the solution of global systems of algebraic equations at each time step.

The mathematical components used in this work are classical. The one-dimensional wave equation, its separation-of-variables solution, the central difference approximation, and the Courant–Friedrichs–Lewy (CFL) stability condition are standard textbook material. The focus of the present study is their systematic integration into a compact and reproducible verification benchmark, in which the exact continuous solution, the closed-form discrete response of the central-difference scheme, and two independent explicit finite element implementations are examined within the same one-dimensional wave propagation problem.

Previous studies have used the one-dimensional wave equation as a benchmark problem for examining stability, convergence, and error behaviour of finite difference schemes [7,8]. In contrast, the present work combines the analytical solution with the closed-form discrete response of the central-difference scheme and explicit finite element models implemented in LS-DYNA [9,10] and PAK-Multiphysics [11]. This connection is important because explicit finite element codes are often used in engineering simulations where no exact solution is available. A simple analytical benchmark with known continuous and discrete reference solutions provides a controlled test problem for assessing the behaviour of the numerical implementation before it is applied to more complex nonlinear dynamic analyses.

The stability and accuracy of explicit schemes are closely related to the spatial and temporal discretization of the governing equations. For hyperbolic problems, such as the one-dimensional wave equation, the CFL stability condition provides a fundamental restriction on the allowable time step [12]. When the CFL restriction is satisfied for the considered explicit scheme, the numerical domain of dependence is consistent with the physical domain of dependence, which is essential for the stable propagation of numerical waves [13–17]. In the present study, the Courant number is kept fixed during the refinement process, so that the spatial and temporal discretizations are refined simultaneously.

In computational mechanics, it is important to distinguish between verification and validation. Code verification concerns whether the numerical implementation correctly solves the intended mathematical formulation, while solution verification concerns the accuracy of a particular numerical solution for a prescribed problem. Model validation, in contrast, is based on comparison with physical experimental data. The present study is positioned within the verification framework: an analytical benchmark solution is used as an exact reference field against which the numerical errors of the finite difference and finite element solutions can be measured directly [18–21].

Although explicit finite element solvers are routinely used in engineering practice, their verification against analytical solutions remains important. Such comparisons are useful not only for commercial software, but also for in-house and academic finite element codes, where benchmark problems with known exact solutions provide a direct way to check the implemented formulation. In the present paper, the verification of PAK-Multiphysics is focused on the explicit linear dynamic response of a one-dimensional axial finite element model with fixed-end boundary conditions, a sinusoidal initial velocity field, a diagonal mass matrix, and central-difference time integration. The same benchmark is also solved in LS-DYNA in order to provide an independent commercial-code comparison.

The aim of this paper is to present an analytical and numerical verification framework for explicit time integration based on the one-dimensional wave equation. First, the convergence of the central difference scheme is examined using a benchmark problem with sinusoidal initial displacement and zero initial velocity. The numerical error is evaluated for a sequence of refined spatial and temporal discretizations at a fixed Courant number in order to confirm the expected second-order convergence along this coupled refinement path.

In the second part of the study, a complementary benchmark problem is introduced, in which the initial displacement is zero while the initial velocity is prescribed as a sinusoidal field. This problem has the same spatial eigenmode as the previous one but differs in the temporal phase of the solution. The use of these two phase-shifted initial conditions provides a stronger verification setting than a single benchmark alone: the first case is suitable for deriving and testing the closed-form discrete central difference response, whereas the second case is directly compatible with explicit finite element verification through prescribed initial nodal velocities.

The main contribution of the paper is the combined use of analytical benchmark solutions, the closed-form discrete central-difference response, the finite-difference/finite-element connection, convergence analysis, and independent explicit finite element verification of one-dimensional wave propagation. In the finite element part of the study, the numerical response is evaluated over several complete oscillation periods and for a sequence of refined finite element meshes. The comparison includes displacement histories, maximum displacement errors, root-mean-square errors, amplitude errors, error-reduction trends under mesh refinement, and a total-energy check. In this way, the study establishes a direct link between the mathematical properties of the wave equation, the CFL stability restriction, the convergence behaviour of the central difference scheme, the corresponding lumped-mass finite element formulation, and practical explicit dynamic finite element simulations.

2. Mathematical Formulation of the Benchmark Problem

The benchmark problem is based on the one-dimensional wave equation defined on the spatial interval $0 < x < L$ and for time $t > 0$, with homogeneous Dirichlet boundary conditions [22]:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \quad (1)$$

where $u(x, t)$ is the axial displacement and c is the wave propagation velocity. The boundary conditions are prescribed as

$$u(0, t) = 0, \quad u(L, t) = 0. \quad (2)$$

For these boundary conditions, the sinusoidal spatial functions represent the eigenmodes of the corresponding spatial problem. The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \dots \quad (3)$$

In this work, only the first mode is considered:

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right), \quad \lambda_1 = \left(\frac{\pi}{L}\right)^2. \quad (4)$$

The corresponding angular frequency is

$$\omega_1 = c\sqrt{\lambda_1} = \frac{\pi c}{L}. \quad (5)$$

Therefore, the first-mode solution can be written in the form

$$u(x, t) = [C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)] \sin\left(\frac{\pi x}{L}\right), \quad (6)$$

where the constants C_1 and C_2 are determined from the prescribed initial conditions. The two benchmark cases considered in this study differ only in these constants and, consequently, in the temporal phase of the response, while the spatial eigenmode remains the same.

The first benchmark problem is defined by a sinusoidal initial displacement and zero initial velocity:

$$u(x, 0) = U_0 \sin\left(\frac{\pi x}{L}\right), \quad \frac{\partial u}{\partial t}(x, 0) = 0. \quad (7)$$

Substitution into the first-mode solution gives $C_1 = U_0$ and $C_2 = 0$. Hence, the corresponding analytical solution is

$$u_1(x, t) = U_0 \sin\left(\frac{\pi x}{L}\right) \cos(\omega_1 t). \quad (8)$$

This case is used for the convergence verification of the central difference discretization, because the exact solution is known at every grid point and time instant.

The second benchmark problem is defined by zero initial displacement and a sinusoidal initial velocity field:

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = V_0 \sin\left(\frac{\pi x}{L}\right). \quad (9)$$

Substitution into the first-mode solution gives $C_1 = 0$ and $C_2 = V_0/\omega_1$. The analytical solution is therefore

$$u_2(x, t) = \frac{V_0}{\omega_1} \sin\left(\frac{\pi x}{L}\right) \sin(\omega_1 t). \quad (10)$$

If the initial velocity amplitude is selected as

$$V_0 = \omega_1 U_0, \quad (11)$$

then the solution becomes

$$u_2(x, t) = U_0 \sin\left(\frac{\pi x}{L}\right) \sin(\omega_1 t). \quad (12)$$

Both benchmark solutions have the same spatial factor, $\sin(\pi x/L)$, but differ in their temporal dependence. The first solution contains $\cos(\omega_1 t)$, while the second contains $\sin(\omega_1 t)$. Thus, the two benchmark problems are phase-shifted variants of the same fundamental wave mode. This property is useful because the first case is well suited for the convergence verification of the central-difference scheme, whereas the second case is convenient for explicit finite element verification because it can be imposed directly through prescribed nodal initial velocities.

3. Central Difference Discretization and Convergence Verification

The first benchmark problem is used to verify the convergence behaviour of the explicit central difference discretization [13–17]. The continuous spatial and temporal domains are replaced by a uniform grid. In the convergence study, the Courant number is kept fixed, so that the spatial and temporal discretizations are refined simultaneously. Therefore, the numerical experiment verifies the convergence behaviour of the method along a coupled refinement path, rather than separating the spatial and temporal error contributions independently.

The nodal coordinates and discrete time instants are defined as

$$x_i = i\Delta x, \quad i = 0, 1, \dots, N, \quad (13)$$

and

$$t_n = n\Delta t, \quad n = 0, 1, 2, \dots \quad (14)$$

The spatial and temporal increments are given by

$$\Delta x = \frac{L}{N}, \quad (15)$$

and Δt , respectively. The numerical approximation of the displacement field at the grid point (x_i, t_n) is denoted by

$$u_i^n \approx u(x_i, t_n). \quad (16)$$

The second derivatives in time and space are approximated using central finite differences [13–17]. At the grid point (x_i, t_n) , the second time derivative is approximated as

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}, \quad (17)$$

while the second spatial derivative is approximated as

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}. \quad (18)$$

These are standard central finite-difference approximations for second derivatives in hyperbolic initial-boundary value problems [13,14]. A compact truncation-error statement follows from Taylor expansions about the point (x_i, t_n) :

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) = \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} + O(\Delta t^2), \quad (19)$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_n) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + O(\Delta x^2). \quad (20)$$

Equations (19) and (20) show that the central finite-difference approximations are second-order consistent in time and space for sufficiently smooth solutions. In the numerical convergence study below, Δx and Δt are refined simultaneously at a fixed Courant number. Consequently, the observed convergence order corresponds to this coupled refinement path. Substitution of Equations (17) and (18) into the wave equation, Equation (1), gives

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}. \quad (21)$$

The stability condition is expressed by introducing the Courant number [12]

$$r = \frac{c\Delta t}{\Delta x}, \quad (22)$$

Equation (21) can be rearranged into the explicit recurrence relation

$$u_i^{n+1} = 2u_i^n - u_i^{n-1} + r^2(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \quad i = 1, 2, \dots, N-1. \quad (23)$$

Equivalently, this expression may be written as

$$u_i^{n+1} = 2(1 - r^2)u_i^n + r^2u_{i+1}^n + r^2u_{i-1}^n - u_i^{n-1}. \quad (24)$$

The explicit recurrence relation is conditionally stable. For the one-dimensional wave equation discretized by the central difference scheme, the CFL stability condition requires [12,13,16]

$$r = \frac{c\Delta t}{\Delta x} \leq 1. \quad (25)$$

Equivalently, the time step must satisfy [12,13,16]

$$\Delta t \leq \frac{\Delta x}{c}. \quad (26)$$

This condition ensures that the numerical domain of dependence includes the physical domain of dependence, which is essential for stable wave propagation in explicit schemes [1,12,13]. The homogeneous boundary conditions are imposed directly at the boundary nodes:

$$u_0^n = 0, \quad u_N^n = 0, \quad n = 0, 1, 2, \dots \quad (27)$$

For the first benchmark problem, the initial displacement field is defined as

$$u_i^0 = U_0 \sin\left(\frac{\pi x_i}{L}\right), \quad i = 0, 1, \dots, N. \quad (28)$$

Since the initial velocity is zero, the first time step must be initialized consistently with the governing equation. Using a Taylor expansion in time and the spatial central finite-difference approximation gives

$$u_i^1 = u_i^0 + \frac{r^2}{2} (u_{i+1}^0 - 2u_i^0 + u_{i-1}^0), \quad i = 1, 2, \dots, N-1. \quad (29)$$

Using $x_i = i\Delta x = iL/N$, the initial displacement in Equation (28) can be written at the grid nodes in the equivalent discrete form

$$u_i^0 = U_0 \sin\left(\frac{\pi i}{N}\right).$$

This is the first discrete sine mode of the spatial mesh, corresponding to the first sinusoidal eigenmode of the continuous boundary-value problem [13,16,22].

$$u_i^n = A_n \sin\left(\frac{\pi i}{N}\right), \quad (30)$$

where A_n is the discrete temporal amplitude. For the discrete sine mode, the following identity holds:

$$\sin\left(\frac{\pi(i+1)}{N}\right) - 2\sin\left(\frac{\pi i}{N}\right) + \sin\left(\frac{\pi(i-1)}{N}\right) = -4\sin^2\left(\frac{\pi}{2N}\right)\sin\left(\frac{\pi i}{N}\right). \quad (31)$$

Substitution of Equation (30) into the central difference recurrence, Equation (24), gives the discrete temporal equation

$$A_{n+1} = 2\left[1 - 2r^2\sin^2\left(\frac{\pi}{2N}\right)\right]A_n - A_{n-1}. \quad (32)$$

Introducing the discrete phase increment θ_N by

$$\cos \theta_N = 1 - 2r^2\sin^2\left(\frac{\pi}{2N}\right), \quad (33)$$

the solution of the recurrence can be written as

$$A_n = U_0 \cos(n\theta_N), \quad (34)$$

because the consistent initialization for zero initial velocity gives $A_0 = U_0$ and $A_1 = U_0 \cos \theta_N$. Hence, the closed-form discrete solution of the central-difference scheme for the first benchmark problem is

$$u_i^n = U_0 \sin\left(\frac{\pi i}{N}\right) \cos(n\theta_N). \quad (35)$$

The quantity θ_N represents the numerical phase increment of the first discrete sine mode over one time step. The corresponding analytical phase increment over the same time step is

$$\theta_{\text{an}} = \omega_1 \Delta t = \frac{\pi c}{L} \Delta t = \frac{\pi r}{N}. \quad (36)$$

The discrete angular frequency associated with the central-difference recurrence may therefore be written as

$$\omega_N = \frac{\theta_N}{\Delta t}. \quad (37)$$

The difference between θ_N and θ_{an} , or equivalently between ω_N and ω_1 , represents the numerical phase or frequency error of the discrete scheme. This is the discrete dispersion effect associated with the central-difference approximation. For increasing N at fixed Courant number r , the numerical phase increment approaches the analytical phase increment, and the closed-form discrete response converges to the analytical first-mode solution.

The closed-form expression for u_i^n represents the solution of the discrete central-difference recurrence for the first benchmark problem. This discrete solution is compared with the analytical solution evaluated at the same grid points and time instants.

To quantify the difference between the analytical and numerical solutions, the point-wise error is defined as

$$e_i^n = u_i^n - u(x_i, t_n). \quad (38)$$

For a fixed final time $T = t_n$, the maximum nodal error is evaluated as

$$E_N = \max_{0 \leq i \leq N} |e_i^n| = \max_{0 \leq i \leq N} |u_i^n - u(x_i, t_n)|. \quad (39)$$

For the first benchmark problem, both the analytical and discrete solutions contain the same spatial sine factor. Therefore, their difference can be written as

$$e_i^n = U_0 [\cos(n\theta_N) - \cos(\omega_1 t_n)] \sin\left(\frac{\pi i}{N}\right). \quad (40)$$

For the even values of N used in this study, the sine factor reaches its maximum absolute value at the middle node, $i = N/2$, where

$$\sin\left(\frac{\pi}{2}\right) = 1. \quad (41)$$

Consequently, for the selected first-mode benchmark, the maximum nodal error is equal to the absolute error at the middle node:

$$E_N = \left| u_{N/2}^n - u\left(\frac{L}{2}, t_n\right) \right|. \quad (42)$$

When the mesh is refined from N to $2N$ elements while keeping the Courant number r constant, both the spatial step Δx and the time step Δt are reduced by a factor of two. The observed order of convergence along this coupled space–time refinement path is estimated from the error ratio as

$$p = \log_2 \left(\frac{E_N}{E_{2N}} \right). \quad (43)$$

For a method that is second-order accurate along this refinement path, the error is expected to decrease approximately by a factor of four:

$$\frac{E_N}{E_{2N}} \approx 4, \quad p \approx 2. \quad (44)$$

This convergence estimate does not separate the spatial and temporal error contributions independently, because the Courant number is held fixed during the refinement process. The following values are selected as benchmark parameters in order to define a simple reference problem with a known analytical solution:

$$L = 10 \text{ m}, \quad c = 5000 \text{ m/s}, \quad U_0 = 1. \quad (45)$$

In this convergence study, U_0 is used as a normalized displacement amplitude. Therefore, the numerical errors reported in this section are normalized with respect to the prescribed unit amplitude. With these values, the analytical solution of the first benchmark problem becomes

$$u(x, t) = \sin\left(\frac{\pi x}{10}\right) \cos(500\pi t). \quad (46)$$

The comparison is performed at the final time

$$T = 8 \cdot 10^{-4} \text{ s}. \quad (47)$$

At this time, the analytical solution is

$$u(x, T) = \sin\left(\frac{\pi x}{10}\right) \cos(0.4\pi). \quad (48)$$

Since

$$\cos(0.4\pi) = 0.309016994, \quad (49)$$

the exact displacement field at the comparison time is

$$u(x, T) = 0.309016994 \sin\left(\frac{\pi x}{10}\right). \quad (50)$$

This fixed comparison time is used only for the closed-form central-difference convergence test in the first benchmark problem. The finite element verification in the following section is performed over several complete oscillation periods.

The convergence study is performed using four spatial discretizations:

$$N = 20, 40, 80, 160. \quad (51)$$

For all meshes, the Courant number is kept constant and equal to

$$r = 0.5. \quad (52)$$

Therefore, for each spatial discretization, the time step is determined from

$$\Delta t = \frac{r\Delta x}{c}. \quad (53)$$

Since $\Delta x = L/N$, refinement of the spatial mesh from N to $2N$ also reduces the time step by the same factor. Thus, the numerical experiment represents a coupled space–time refinement study at a fixed Courant number.

Before the convergence ratios are evaluated, the analytical and closed-form discrete central difference values at the middle node are compared directly. For the middle node, $i = N/2$, the spatial sine factor is equal to one. Therefore, the analytical and discrete solutions reduce to

$$u\left(\frac{L}{2}, T\right) = \cos(\omega_1 T), \quad u_{N/2}^n = \cos(n\theta_N). \quad (54)$$

Since $U_0 = 1$, these values represent normalized displacement amplitudes. For the first-mode benchmark considered here, the maximum nodal error coincides with the absolute middle-node error, as shown above. The corresponding analytical and closed-form central difference values are compared in Table 1, together with the resulting maximum nodal error E_N .

Table 1. Middle-node comparison between the analytical and closed-form central-difference solutions at $T = 8 \cdot 10^{-4}$ s.

N	n	$u(L/2, T)$	$u_{N/2}^n$	E_N
20	16	0.30901699	0.30993873	$9.2173 \cdot 10^{-4}$
40	32	0.30901699	0.30924739	$2.3039 \cdot 10^{-4}$
80	64	0.30901699	0.30907459	$5.7596 \cdot 10^{-5}$
160	128	0.30901699	0.30903139	$1.4399 \cdot 10^{-5}$

The convergence results are summarized in Table 2. The first column gives the number of spatial elements N , while Δx and Δt denote the corresponding spatial and temporal increments. The column n represents the number of time steps required to reach the comparison time T . The quantity E_N is the maximum nodal error defined above. The ratio E_N/E_{2N} compares the errors obtained on two successive meshes, and p denotes the observed convergence order along the coupled space–time refinement path. The number of time steps n is selected so that $t_n = T$ for each discretization.

Table 2. Convergence verification of the central-difference solution for the first benchmark problem at $T = 8 \cdot 10^{-4}$ s.

N	Δx [m]	Δt [s]	n	E_N	E_N/E_{2N}	p
20	0.5000	$5.00 \cdot 10^{-5}$	16	$9.2173 \cdot 10^{-4}$	4.0007	2.0002
40	0.2500	$2.50 \cdot 10^{-5}$	32	$2.3039 \cdot 10^{-4}$	4.0001	2.0001
80	0.1250	$1.25 \cdot 10^{-5}$	64	$5.7596 \cdot 10^{-5}$	4.0000	2.0000
160	0.0625	$6.25 \cdot 10^{-6}$	128	$1.4399 \cdot 10^{-5}$	—	—

The results in Table 2 show that the maximum nodal error decreases by approximately a factor of four after each mesh refinement. Consequently, the observed convergence order remains very close to $p = 2$ for all refinement levels. Since the Courant number is kept fixed, this result confirms second-order convergence along the coupled space–time refinement path. This behaviour is consistent with the second-order truncation-error estimates of the central finite-difference approximations in space and time. Therefore, the closed-form discrete central-difference solution reproduces the analytical first-mode solution with the expected convergence behaviour for the selected benchmark problem.

Relation Between the Finite Difference and Finite Element Formulations

For a uniform one-dimensional axial finite element mesh with linear two-node elements, element length $h = \Delta x$, cross-sectional area A , Young's modulus E , and density ρ , the semi-discrete finite element equation can be written as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \quad (55)$$

For an interior node and a diagonal lumped mass matrix, the nodal mass is

$$m_i = \rho Ah, \quad (56)$$

while the stiffness contribution of the neighbouring elements gives

$$\frac{EA}{h}(2u_i - u_{i-1} - u_{i+1}). \quad (57)$$

Therefore, the semi-discrete finite element equation at an interior node becomes

$$\rho Ah \ddot{u}_i + \frac{EA}{h}(2u_i - u_{i-1} - u_{i+1}) = 0. \quad (58)$$

After division by ρAh , this equation reduces to

$$\ddot{u}_i = \frac{E}{\rho} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \quad (59)$$

Since the longitudinal wave velocity is $c = \sqrt{E/\rho}$, the finite element equation can be written as

$$\ddot{u}_i = c^2 \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}. \quad (60)$$

This is identical to the semi-discrete equation obtained from the central spatial difference approximation of the one-dimensional wave equation. Consequently, for a uniform mesh, linear axial finite elements, and a lumped mass matrix, the spatial part of the finite element formulation is equivalent to the central finite difference operator derived above. For the first discrete sine mode,

$$u_i = \sin\left(\frac{\pi i}{N}\right), \quad (61)$$

the corresponding semi-discrete finite element angular frequency is

$$\Omega_N = \frac{2c}{\Delta x} \sin\left(\frac{\pi}{2N}\right). \quad (62)$$

Using central difference time integration, the associated time-discrete phase increment satisfies

$$\cos \theta_N = 1 - \frac{1}{2} \Omega_N^2 \Delta t^2. \quad (63)$$

Since $r = c\Delta t/\Delta x$, this expression becomes

$$\cos \theta_N = 1 - 2r^2 \sin^2\left(\frac{\pi}{2N}\right), \quad (64)$$

which is the same discrete phase relation obtained for the central-difference scheme. This establishes the analytical connection between the finite difference recurrence derived above and the lumped-mass finite element formulation for the present one-dimensional benchmark problem. Therefore, similar numerical behaviour is expected when the same mesh spacing, material wave speed, boundary conditions, and central difference time integration are used.

4. Explicit FEM Verification in LS-DYNA and PAK-Multiphysics

The second benchmark problem, defined by zero initial displacement and a sinusoidal initial velocity field, is used for explicit finite element verification. This benchmark is suitable for finite element implementation because the initial condition can be prescribed

directly as nodal velocities. The objective of this section is not to compare general-purpose finite element software packages, but to verify whether two independent explicit finite element implementations reproduce the same analytical one-dimensional wave response under equivalent geometrical, material, boundary, and initial conditions.

The LS-DYNA model was prepared using the standard keyword-based explicit dynamic analysis framework documented in the LS-DYNA theory and keyword manuals [9,10]. The PAK-Multiphysics model was prepared using the in-house academic finite element software PAK-Multiphysics, developed for the solution of coupled multiphysics problems [11]. The numerical simulations were performed using LS-DYNA within the ANSYS 2022 R2 software environment and PAK-Multiphysics version 8.00. In the present benchmark, LS-DYNA is used as an independent commercial-code comparison, whereas PAK-Multiphysics is used as a finite element implementation closely aligned with the lumped-mass central-difference formulation discussed in the previous section.

For consistency with the LS-DYNA and PAK-Multiphysics finite element models, the benchmark parameters are expressed in the millimetre–tonne–second unit system. The rod length is

$$L = 10000 \text{ mm.}$$

The material parameters are selected so that the longitudinal wave speed is

$$c = 5 \cdot 10^6 \text{ mm/s,}$$

which corresponds to $c = 5000 \text{ m/s}$. For a linearly elastic one-dimensional rod, the longitudinal wave speed is

$$c = \sqrt{\frac{E}{\rho}},$$

where E is Young's modulus and ρ is the mass density [23–25]. The adopted material parameters are

$$E = 195000 \text{ N/mm}^2, \quad \rho = 7.8 \cdot 10^{-9} \text{ tonne/mm}^3,$$

which gives

$$c = \sqrt{\frac{195000}{7.8 \cdot 10^{-9}}} = 5 \cdot 10^6 \text{ mm/s.}$$

The same geometrical and material basis is used in both finite element models. The cross-sectional area of the one-dimensional finite elements was set to $A = 1 \text{ mm}^2$ in both the PAK-Multiphysics and LS-DYNA models. The rod is discretized using uniform meshes with

$$N = 20, 40, 80, 160$$

one-dimensional two-node axial elements. In both PAK-Multiphysics and LS-DYNA, these elements were used with a diagonal, i.e., lumped, mass matrix in the explicit central-difference analyses. The spatial step is

$$\Delta x = \frac{L}{N}.$$

The end nodes of the rod are fixed in the axial direction, corresponding to homogeneous displacement boundary conditions. The internal nodes are allowed to move only in the axial direction. No damping and no mass scaling are introduced in the numerical models. In LS-DYNA, a resultant truss beam formulation is used through `*SECTION_BEAM` with `ELFORM = 3`. In PAK-Multiphysics, the corresponding one-dimensional axial rod

element with a diagonal mass matrix is used. In both models, the relevant active degree of freedom is the axial displacement of the rod. The time step is selected according to the fixed Courant number

$$r = \frac{c\Delta t}{\Delta x} = 0.5,$$

so that

$$\Delta t = \frac{r\Delta x}{c}.$$

The final simulation time is

$$T_{\text{end}} = 0.02 \text{ s},$$

which corresponds to five periods of the first longitudinal mode, since

$$T_1 = \frac{2\pi}{\omega_1} = 0.004 \text{ s}.$$

The mesh and time-step parameters used in the refinement study are summarized in Table 3.

Table 3. Mesh and time-step parameters used in the explicit finite element verification.

N	Δx [mm]	Δt [s]	Number of Time Steps	Midspan Node
20	500.0	$5.00 \cdot 10^{-5}$	400	11
40	250.0	$2.50 \cdot 10^{-5}$	800	21
80	125.0	$1.25 \cdot 10^{-5}$	1600	41
160	62.5	$6.25 \cdot 10^{-6}$	3200	81

For the second benchmark problem, the angular frequency of the first mode is

$$\omega_1 = \frac{\pi c}{L} = \frac{\pi \cdot 5 \cdot 10^6}{10000} = 500\pi \text{ s}^{-1}.$$

The displacement amplitude is selected as

$$U_0 = 1 \text{ mm}.$$

Therefore, the corresponding amplitude of the initial velocity field is

$$V_0 = \omega_1 U_0 = 500\pi \cdot 1 = 1570.796 \text{ mm/s}.$$

The prescribed initial velocity field is

$$v_x(x, 0) = 1570.796 \sin\left(\frac{\pi x}{10000}\right) \text{ mm/s}.$$

This velocity field is assigned to the finite element nodes according to their spatial coordinates. The response is evaluated at the midspan node, where the sinusoidal spatial mode reaches its maximum value. For all considered meshes, this point corresponds to

$$x = \frac{L}{2} = 5000 \text{ mm},$$

and therefore

$$\sin\left(\frac{\pi x}{10000}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

The analytical midspan displacement response is therefore

$$u_{\text{an}}(5000, t) = \sin(500\pi t) \text{ mm.}$$

The finite element model used for the benchmark is shown schematically in Figure 1. The figure illustrates the fixed end nodes, the active axial displacement degree of freedom, the node numbering convention, and the prescribed sinusoidal initial velocity field. For clarity, the schematic is illustrative and shown for a coarse discretization, whereas the numerical verification is performed for the four meshes listed in Table 3.

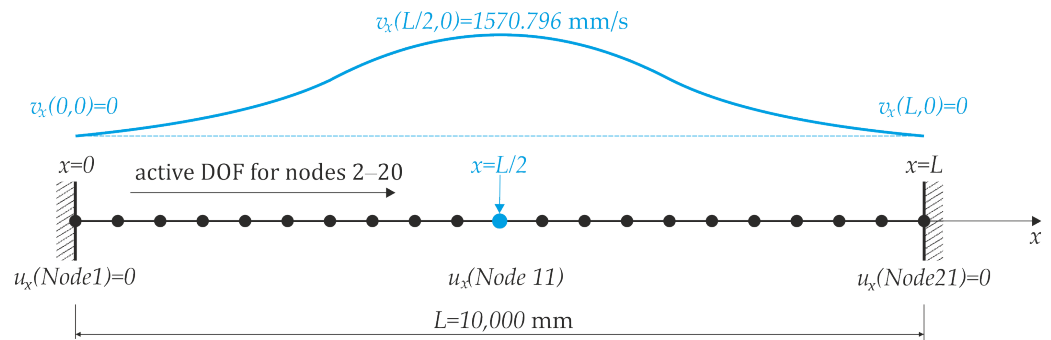


Figure 1. Schematic finite element discretization of the one-dimensional rod benchmark model with fixed end nodes, active axial displacement degree of freedom, node numbering convention, and prescribed sinusoidal initial velocity field.

The midspan displacement histories obtained from the analytical solution, LS-DYNA, and PAK-Multiphysics are first compared for the refined mesh with $N = 160$. This comparison is performed over the complete time interval $0 \leq t \leq 0.02$ s, corresponding to five periods of the first longitudinal mode. The analytical solution is evaluated at the same output times as the numerical results, using the midspan response $u_{\text{an}}(5000, t) = \sin(500\pi t)$.

Although the displacement histories in Figure 2 show very close agreement between the analytical and numerical responses, a quantitative error assessment is required to evaluate the behaviour of the models under mesh and time-step refinement. The comparison is therefore based on error measures computed over the complete simulation interval $0 \leq t \leq 0.02$ s.

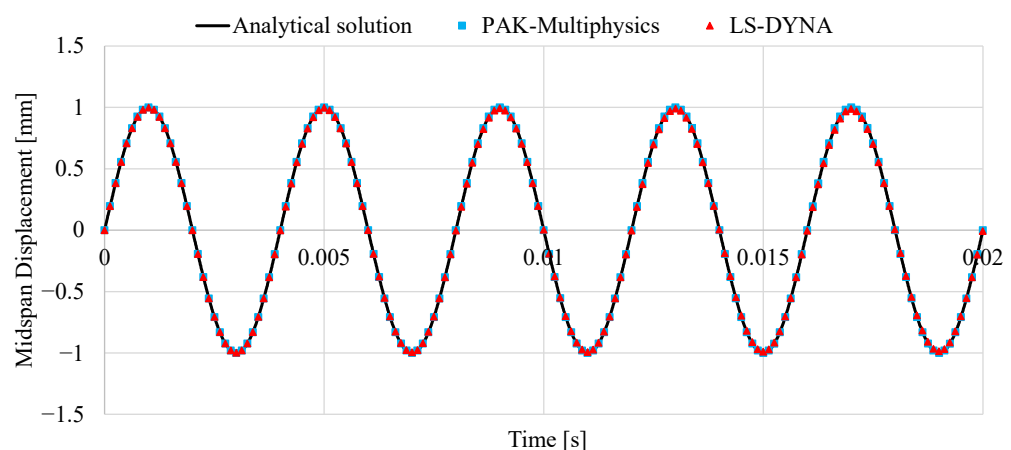


Figure 2. Comparison of the analytical, PAK-Multiphysics, and LS-DYNA midspan displacement histories for the refined mesh with $N = 160$ over five periods of the first longitudinal mode.

For each mesh, the midspan displacement error is defined at the numerical output times as

$$e_j = u_{\text{num}}(5000, t_j) - u_{\text{an}}(5000, t_j),$$

where u_{num} denotes the LS-DYNA or PAK-Multiphysics displacement response and u_{an} is the analytical solution evaluated at the same time t_j . The maximum absolute displacement error over the complete time interval is defined as

$$E_{\text{max}} = \max_j |e_j|.$$

The root mean square error over the same interval is computed as

$$E_{\text{RMS}} = \sqrt{\frac{1}{M} \sum_{j=1}^M e_j^2},$$

where M is the number of numerical output points. In addition, the relative RMS error is defined with respect to the analytical displacement history as

$$E_{\text{RMS,rel}} = \sqrt{\frac{\sum_{j=1}^M e_j^2}{\sum_{j=1}^M u_{\text{an}}^2(5000, t_j)}} \cdot 100\%.$$

To assess whether the numerical models preserve the amplitude of the oscillatory response, the numerical amplitude is evaluated as

$$A_{\text{num}} = \frac{\max_j u_{\text{num}}(5000, t_j) - \min_j u_{\text{num}}(5000, t_j)}{2}.$$

The corresponding relative amplitude error is then defined as

$$E_A = \frac{|A_{\text{num}} - U_0|}{U_0} \cdot 100\%,$$

where $U_0 = 1$ mm is the prescribed displacement amplitude of the analytical solution. The error measures are evaluated for all four meshes and for both explicit finite element models. The analytical displacement response is evaluated at the same output times as the corresponding numerical solution, so that the computed errors reflect the actual time histories produced by each solver. The resulting maximum absolute errors, RMS errors, relative RMS errors, and amplitude errors are summarized in Table 4.

Table 4. Error measures for the midspan displacement histories over the complete simulation interval $0 \leq t \leq 0.02$ s.

N	Model	E_{max} [mm]	E_{RMS} [mm]	$E_{\text{RMS,rel}}$ [%]	E_A [%]
20	PAK-Multiphysics	0.024275	0.010044	1.4222	0.18
20	LS-DYNA	0.109536	0.047088	6.6750	1.0284
40	PAK-Multiphysics	0.006064	0.002507	0.3547	0.04
40	LS-DYNA	0.056498	0.024257	3.4343	0.5654
80	PAK-Multiphysics	0.001515	0.000626	0.0886	0.01
80	LS-DYNA	0.028754	0.012344	1.7465	0.2956
160	PAK-Multiphysics	0.000383	0.000156	0.0221	0.00
160	LS-DYNA	0.014514	0.006230	0.8813	0.1510

The results in Table 4 show that both finite element models reproduce the analytical midspan response with decreasing error as the mesh and time step are refined. For PAK-Multiphysics, the maximum absolute error decreases from $2.4275 \cdot 10^{-2}$ mm for $N = 20$ to

$3.83 \cdot 10^{-4}$ mm for $N = 160$. The RMS and relative RMS errors show the same systematic reduction, and the amplitude error becomes negligible for the finest mesh.

The LS-DYNA results also show a monotonic reduction of the error measures under mesh and time-step refinement. The maximum absolute error decreases from $1.09536 \cdot 10^{-1}$ mm for $N = 20$ to $1.4514 \cdot 10^{-2}$ mm for $N = 160$, while the relative RMS error decreases from 6.6750% to 0.8813%. These results indicate a consistent error-reduction trend for the independent commercial-code model over the complete time-history comparison.

The formal convergence behaviour of the central-difference discretization is assessed in Section 3 using the closed-form discrete solution. The PAK-Multiphysics results are used to examine the corresponding finite element benchmark model under the same mesh-refinement sequence and fixed Courant number. The LS-DYNA results are used as an independent commercial-code comparison for the same analytical benchmark and are interpreted in terms of displacement-history agreement, bounded energy behaviour, and error-reduction trends under mesh refinement.

To visualize the refinement behaviour more clearly, the maximum absolute midspan displacement error is plotted as a function of the number of finite elements in Figure 3. The figure summarizes the error-reduction trend for both explicit finite element models over the complete five-period simulation interval.

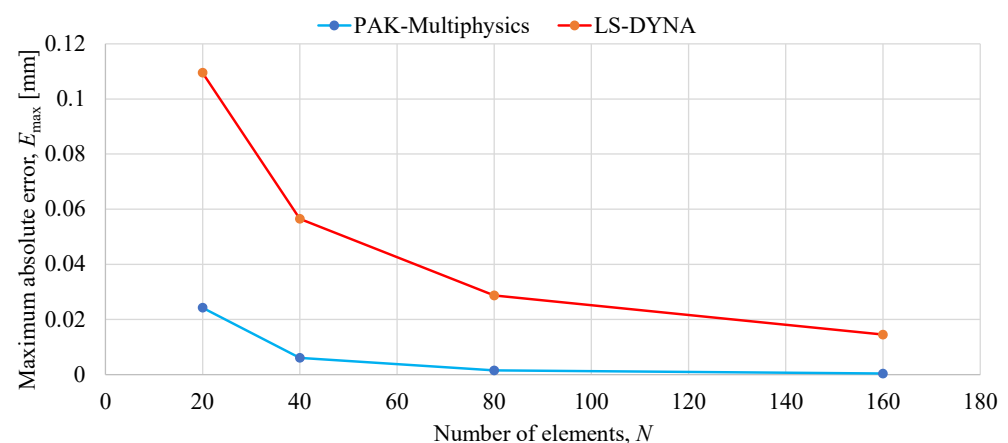


Figure 3. Maximum absolute midspan displacement error as a function of mesh refinement for PAK-Multiphysics and LS-DYNA over the interval $0 \leq t \leq 0.02$ s.

Figure 3 confirms the error-reduction trend observed in Table 4. For PAK-Multiphysics, the maximum error decreases rapidly with mesh refinement, which is consistent with the lumped-mass central-difference formulation discussed previously. The LS-DYNA results also exhibit a systematic decrease of the maximum error as the mesh and time step are refined.

To further examine the temporal distribution of the numerical error, the midspan displacement error histories for the refined mesh with $N = 160$ are shown in Figure 4. The error is computed according to $e_j = u_{\text{num}}(5000, t_j) - u_{\text{an}}(5000, t_j)$ over the complete five-period simulation interval.

Figure 4 shows that the PAK-Multiphysics error remains close to zero over the complete simulation interval for the refined mesh. The LS-DYNA error also remains bounded, with an oscillatory component over the five-period interval. This plot provides a more sensitive view of the small differences that are not clearly visible in the displacement histories shown in Figure 2, while the overall benchmark response is still reproduced with good agreement.

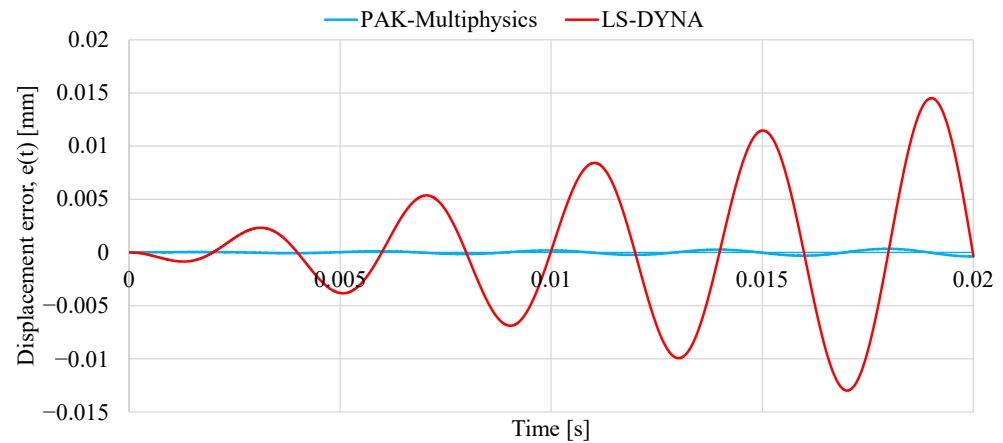


Figure 4. Midspan displacement error histories for PAK-Multiphysics and LS-DYNA for the refined mesh with $N = 160$ over the interval $0 \leq t \leq 0.02$ s.

As an additional check of the explicit dynamic response, the total energy history was examined for the LS-DYNA model. Since the benchmark problem has fixed boundaries, no external loading after the initial condition, and no prescribed damping, the total mechanical energy should remain bounded and approximately conserved in the numerical solution. The relative total-energy variation plotted in Figure 5 was defined as

$$\Delta E_{\text{rel}}(t) = \frac{|E_{\text{tot}}(t) - E_{\text{tot}}(0)|}{E_{\text{tot}}(0)} \times 100\%,$$

where $E_{\text{tot}}(t)$ is the total mechanical energy at time t , and $E_{\text{tot}}(0)$ is the initial total mechanical energy. Therefore, the total energy response provides an additional indicator that the observed displacement errors are not associated with unstable energy growth or numerical damping.

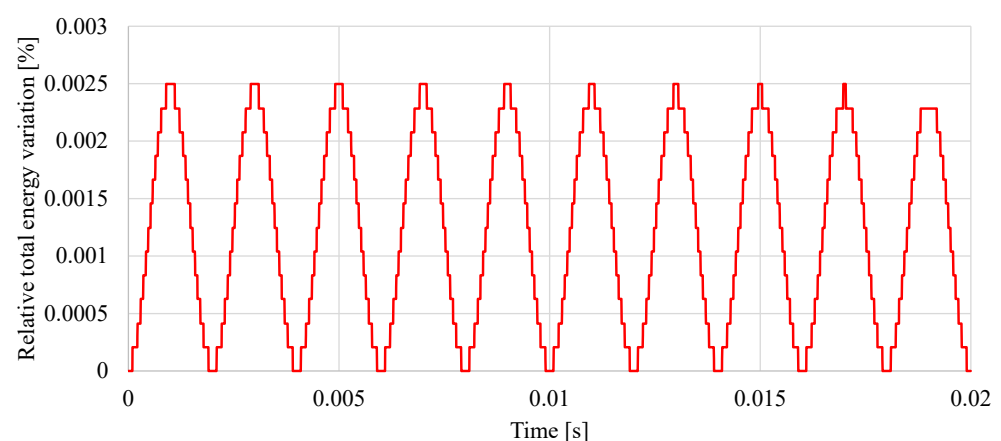


Figure 5. Relative total energy variation of the LS-DYNA model for the refined mesh with $N = 160$ over the interval $0 \leq t \leq 0.02$ s.

Figure 5 shows that the LS-DYNA total energy remains bounded over the complete five-period simulation interval. No systematic energy growth or decay is observed for the adopted stable time step. This indicates that the larger LS-DYNA displacement errors reported above are not associated with an unstable energy response, and are more

likely related to code-specific finite element and time-integration details for the present benchmark configuration.

Overall, the explicit finite element verification is based on complete displacement and error histories over five oscillation periods. The PAK-Multiphysics results show very close agreement with the analytical solution and a systematic reduction of the error measures under coupled mesh and time-step refinement, which is consistent with the lumped-mass central-difference formulation. The LS-DYNA results reproduce the same analytical benchmark response with bounded energy variation and decreasing displacement error under refinement, providing an independent commercial-code comparison. Together with the closed-form central-difference analysis presented in the previous section, these results provide a reproducible verification framework for one-dimensional explicit wave propagation models.

5. Conclusions

This paper presented an analytical and numerical verification framework for explicit time integration procedures applied to the one-dimensional wave equation. Two benchmark problems with homogeneous Dirichlet boundary conditions and the same first sinusoidal spatial mode were considered. The first benchmark problem, defined by sinusoidal initial displacement and zero initial velocity, was used to analyse the closed-form discrete central-difference response and the convergence behaviour of the discretization. The second benchmark problem, defined by zero initial displacement and sinusoidal initial velocity, was used for explicit finite element verification in LS-DYNA and PAK-Multiphysics.

The central-difference analysis showed that the scheme is second-order consistent in both time and space, with leading truncation-error terms of order $O(\Delta t^2)$ and $O(\Delta x^2)$. Under the fixed Courant-number refinement path used in this study, the maximum nodal error decreased approximately by a factor of four when the spatial and temporal discretizations were refined by a factor of two. The observed convergence order remained very close to $p = 2$, confirming the expected second-order behaviour of the central-difference discretization for the considered analytical benchmark problem.

The explicit finite element verification was performed using complete displacement and error histories over five periods of the first longitudinal mode. Four meshes, $N = 20, 40, 80, 160$, were analysed with coupled mesh and time-step refinement at fixed Courant number. The PAK-Multiphysics results showed very close agreement with the analytical solution and a systematic reduction of the error measures under refinement. The maximum absolute midspan displacement error decreased from $2.4275 \cdot 10^{-2}$ mm for $N = 20$ to $3.83 \cdot 10^{-4}$ mm for $N = 160$, while the relative RMS error decreased from 1.4222% to 0.0221%. This behaviour is consistent with the lumped-mass central-difference formulation discussed in the paper.

The LS-DYNA results also reproduced the analytical benchmark response with decreasing displacement error under mesh and time-step refinement. The maximum absolute error decreased from $1.09536 \cdot 10^{-1}$ mm for $N = 20$ to $1.4514 \cdot 10^{-2}$ mm for $N = 160$, while the relative RMS error decreased from 6.6750% to 0.8813%. The total energy history remained bounded, with no systematic energy growth or decay for the adopted stable time step. Therefore, the LS-DYNA model provides an independent commercial-code comparison for the same analytical benchmark, while the formal convergence behaviour of the central-difference discretization is evaluated using the closed-form discrete solution, with the PAK-Multiphysics results providing a closely aligned finite element benchmark comparison.

The proposed benchmark framework is simple, reproducible, and transparent. Its main value is that it connects the analytical solution of the one-dimensional wave equation, the closed-form discrete central-difference response, and explicit finite element implementa-

tions under controlled initial-boundary conditions. The use of two phase-shifted analytical solutions makes it possible to verify both the convergence of the central-difference discretization and the response of explicit finite element models. The framework can be extended to more complex benchmark problems involving higher modes, nonuniform discretizations, different boundary conditions, multidimensional wave propagation, and nonlinear constitutive models.

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