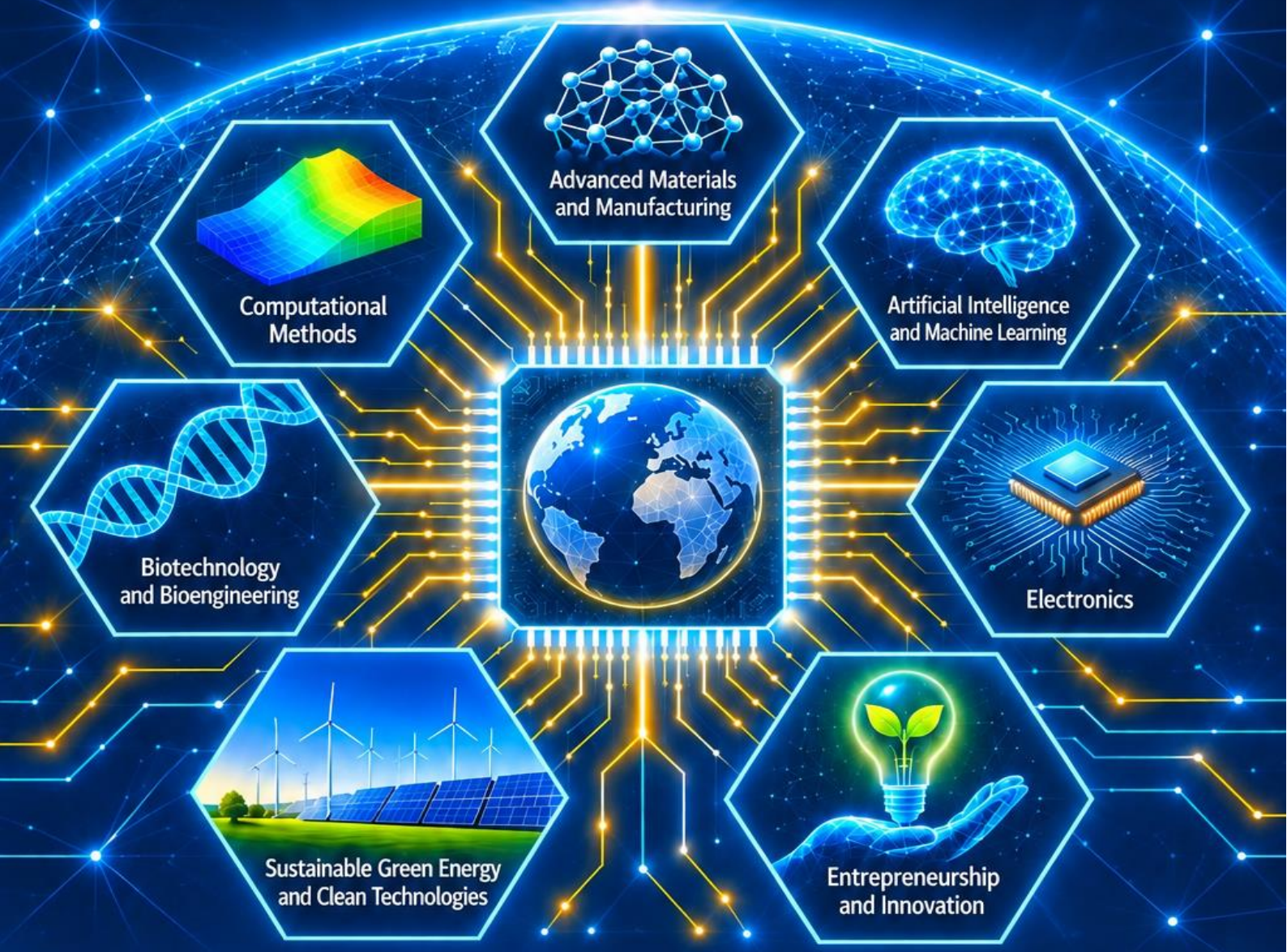
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On the construction of multiple orthogonal polynomials with nearly diagonal multi-indices with respect to bilinear forms

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Abstract

Multiple orthogonal polynomials generalize ordinary orthogonal polynomials by satisfying orthogonality conditions simultaneously with respect to $r \in \mathbb{N}$ weight functions. In this paper, we present a method for constructing multiple orthogonal polynomials which are orthogonal with respect to r bilinear forms, with particular emphasis on nearly diagonal multi-indices. These polynomials play an essential role in the construction of a set of anti-Gaussian quadrature formulas for the optimal set of quadrature formulas in Borges' sense. The main contribution is based on the connection between these polynomials and multiple orthogonal polynomials which are orthogonal with respect to a corresponding system of r weight functions, which arise in the construction of the optimal set of quadrature formulas in Borges' sense. The results provide a unified framework that connects the theory of multiple orthogonality with respect to bilinear forms to its classical counterpart involving weight functions, offering new insight into the structure of the associated polynomial systems in the nearly diagonal setting. Several numerical examples are included to illustrate the obtained results.

Keywords: weight functions; multiple orthogonal polynomials; anti-Gaussian quadrature rules; optimal set of quadrature rules in Borges' sense

1. Introduction

An important aspect of the theory of orthogonal polynomials is their interpretation as special functions, often expressible as terminating hypergeometric series. Over the twentieth century, substantial effort has been devoted to their classification and characterization. More recently, increased attention has been given to multiple orthogonal polynomials (MOPs), which arise naturally in simultaneous rational approximation, particularly in Hermite–Padé approximation of systems of r functions. In [1] we considered a class of MOPs defined through orthogonality relations with respect to a certain bilinear forms. Here, we are giving a recapitulation of results that presents in [1, 2].

Let $r \geq 1$ be an integer and let $w_1, w_2, \ldots, w_r$ be r weight functions on the real line such that the support of each w_k is a subset of an interval E_k. Vector $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ is called a *multi-index*, where $n_1, n_2, \ldots, n_r$ are nonnegative integers, and its length is $|\vec{n}| = n_1 + n_2 + \ldots + n_r$. MOPs can be classified into two types (see [3]).

Type I MOP is an r-vector $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$, where each $A_{\vec{n},k}$ is a polynomial of degree $n_k - 1$, satisfying the orthogonality conditions

$$\sum_{k=1}^{r} \int_{E_k} A_{\vec{n},k}(x) x^j w_k(x) dx = 0, \quad j = 0, \ldots, |\vec{n}| - 2; \quad \text{with} \quad \sum_{k=1}^{r} \int_{E_k} A_{\vec{n},k}(x) x^{|\vec{n}|-1} w_k(x) dx = 1.$$

Type II MOP is the monic polynomial $\hat{P}_n$ of degree n that satisfies the orthogonality conditions

$$\int_{E_k} \hat{P}_{\vec{n}}(x) x^j w_k(x) dx = 0, \quad j = 0, \ldots, n_k - 1; \quad k = 1, \ldots, r.$$

Orthogonal polynomials on the real line are known to satisfy a three-term recurrence relation (see [4–6]). MOPs, in turn, satisfy recurrence relations of higher but finite order. In this paper, our attention is focused on nearly diagonal multi-indices. Let $m \in \mathbb{N}$ and write $m = lr + j$, where $l = \left[\frac{m}{r}\right]$ and $0 \leq j < r$. The corresponding nearly diagonal multi-index $\vec{s}(m)$ is defined by

$$\vec{s}(m) = (\underbrace{l+1, \ldots, l+1}_{j \text{ times}}, \underbrace{l, \ldots, l}_{r-j \text{ times}}).$$

Type II MOPs associated with nearly diagonal multi-indices satisfy a recurrence of order $r+1$. Let us denote the corresponding type II MOP by $\hat{P}_m = \hat{P}_{\vec{s}(m)}$. Then, the following recurrence relation holds:

$$\boldsymbol{x\hat{P}_m(x) = \hat{P}_{m+1}(x) + \sum_{i=0}^{r} \hat{a}_{m,r-i} \hat{P}_{m-i}(x), \quad m \geq 0,} \tag{1}$$

with initial conditions $\hat{P}_0(x) = 1$ and $\hat{P}_i(x) = 0$, for $i = -1, \ldots, -r$ (see [6]).

2. Multiple orthogonal polynomials with respect to certain bilinear forms

Let $(w_1, \ldots, w_r)$ be an AT system of r weight functions on the real line, all supported on the same interval E. We introduce the following notation for the corresponding inner products:

$$\boldsymbol{\langle f, g \rangle_k = I_k(fg), \quad k = 1, \ldots, r,} \tag{2}$$

where $I_k f = \int_E f(x) w_k(x) \mathrm{d}x, \quad k = 1, \ldots, r.$

Let $G_n^{(k)}(f) = \sum_{i=1}^{n} \hat{A}_{k,i} f(\hat{x}_i)$, $k = 1, \ldots, r$, denote the quadrature rules from the optimal set of quadrature rules in Borges' sense, and let $H_{n+1}^{(k)}$, $k = 1, \ldots, r$, denote the corresponding anti-Gaussian quadrature rules. From the identity $I_k p - H_{n+1}^{(k)} p = -\left(I_k p - G_n^{(k)} p\right)$, valid for all polynomials p of degree at most n, we obtain $H_{n+1}^{(k)} p = \left(2I_k - G_n^{(k)}\right) p$, which gives rise to consider the following bilinear forms:

$$\boldsymbol{(f, g)_k = \left(2I_k - G_n^{(k)}\right)(fg), \quad k = 1, \ldots, r.} \tag{3}$$

Now we introduce two types of MOPs with respect to the bilinear forms (3).

A type I MOP is an r-tuple of polynomials $(B_{\vec{n},1}, \ldots, B_{\vec{n},r})$, where each $B_{\vec{n},k}$ is of degree at most $n_k - 1$, satisfying the orthogonality conditions

$$\sum_{k=1}^{r}\left(B_{\vec{n},k},x^{i}\right)_{k}=0,\quad i=0,\ldots,|\vec{n}|-2,\quad \text{with}\quad \sum_{k=1}^{r}\left(B_{\vec{n},k},x^{|\vec{n}|-1}\right)_{k}=1.$$

A type II MOP is the monic polynomial $P_{\vec{n}}$ of degree $|\vec{n}|$ satisfying the orthogonality conditions

$$\left(P_{\vec{n}},x^{i}\right)_{k}=0,\quad i=0,\ldots,n_{k}-1,\ \ k=1,\ldots,r.$$

3. The nearly diagonal multi-indices

Since we restrict our attention to nearly diagonal multi-indices, we introduce, for the sake of simplicity, the notation $P_n = P_{\vec{n}}$ for the corresponding type II MOPs with respect to (3), where $n = |\vec{n}|$. The results presented here can be found in [1, 2].

Theorem 5. *The type II MOPs with respect to the bilinear forms* (3) *with nearly diagonal multi-indices satisfy the following recurrence relation*

$$\boldsymbol{xP_m(x)=P_{m+1}(x)+\sum_{i=0}^{r}a_{m,r-i}P_{m-i}(x),\quad m\geq 0,}\qquad \textbf{(4)}$$

with initial conditions $P_0(x)=1$, $P_i(x)=0$ *for* $i=-1,\ldots,-r$.

Assuming that the bilinear forms (3) satisfy the periodicity condition $(\cdot,\cdot)_{i+Mr}=(\cdot,\cdot)_i$, $M\in\mathbb{Z}$, the following result holds (see [2]).

Theorem 1. Let $m=lr+j$, where $l=\left[\frac{m}{r}\right]$ and $j\in\{0,\ldots,r-1\}$. The recurrence coefficients in (4), for $i=1,\ldots,r$ and $m\geq 0$, can be expressed as

$$a_{m,0}=\frac{\left(xP_m,P_{\left[\frac{m-r}{r}\right]}\right)_{j+1}}{\left(P_{m-r},P_{\left[\frac{m-r}{r}\right]}\right)_{j+1}},\qquad a_{m,i}=\frac{\left(xP_m-\sum_{v=0}^{i-1}a_{m,v}P_{m-r+v},P_{\left[\frac{m-r+i}{r}\right]}\right)_{j+1+i}}{\left(P_{m-r+i},P_{\left[\frac{m-r+i}{r}\right]}\right)_{j+1+i}},$$

Theorem 2. Let $(w_1,\ldots,w_r)$ be a system of r weight functions on the real line. For type II MOPs with nearly diagonal multi-indices, orthogonal with respect to the bilinear forms (3), the recurrence coefficients in (4) can be expressed as

$$\begin{aligned}&a_{m,i}=\hat{a}_{m,i},\quad i=0,\ldots,r,\ \ m<n,\\&a_{n,i}=2\hat{a}_{n,i},\quad i=0,\ldots,r-1,\quad a_{n,r}=\hat{a}_{n,r},\end{aligned}$$

and $P_i(x)=\hat{P}_i(x), i\leq n$, where $\hat{a}_{m,i}$ are the recurrence coefficients in (1) corresponding to the type II MOPs $\hat{P}_i$, defined with respect to the inner products (2).

4. Numerical examples

We consider three weight functions $w_1(x) = (1-x)^{1/2}(1+x)^{1/3}$, $w_2(x) = (1-x)^{1/2}(1+x)^{1/4}$, and $w_3(x) = (1-x)^{1/2}(1+x)$ on the interval $[-1,1]$. The zeros of the corresponding type II MOPs $P_{(3,2,2)}(x)$ and $P_{(5,5,4)}(x)$, defined with respect to the bilinear forms (3), are listed below.

For $P_{(3,2,2)}(x)$: $-1.0294522350, -0.9307665589, -0.6782377807, -0.2590348297, -0.7729332667, 0.6861526953, 0.9505300654$.
For $P_{(5,5,4)}(x)$: $-1.0332207961, -0.9954369264, -.09650839303, -0.8927894073, -0.7729332667, -0.6056447260, -0.3967282671, -0.1572290866, -0.0976349660, 0.3499657997, 0.5809409654, 0.7726731243, 0.9100603085, 0.9832600281$.

4. Conclusion

We present a stable numerical method for constructing MOPs with respect to certain bilinear forms. The approach leverages previously known MOPs with respect to inner products, thereby reducing the need for extensive additional computations in generating new ones.

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