mathematics

Article

Some Results on $(s-q)$-Graphic Contraction Mappings in b-Metric-Like Spaces

Manuel De la Sen [1,*], Nebojša Nikolić [2], Tatjana Došenović [3], Mirjana Pavlović [4] and Stojan Radenović [5]

1 Institute of Research and Development of Porocesses, University of the Basque, 48080 Bilbao, Spain
2 Faculty of Organizational Sciences, University of Belgrade, 11000 Belgrade, Serbia; nebojsa.nikolic@fon.bg.ac.rs
3 Faculty of Technology, University of Novi Sad, Bulevar cara Lazara 1, 21000 Novi Sad, Serbia; tatjanad@tf.uns.ac.rs
4 Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujeva, Serbia; mpavlovic@kg.ac.rs
5 Faculty of Mechanical Enginering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia; radens@beotel.rs or sradenovic@mas.bg.ac.rs
* Correspondence: manuel.delasen@ehu.eus

Received: 15 November 2019; Accepted: 2 December 2019; Published: 4 December 2019

Abstract: In this paper we consider $(s-q)$-graphic contraction mapping in b-metric like spaces. By using our new approach for the proof that a Picard sequence is Cauchy in the context of b-metric-like space, our results generalize, improve and complement several approaches in the existing literature. Moreover, some examples are presented here to illustrate the usability of the obtained theoretical results.

Keywords: b-metric space; b-metric-like space; general contractive mappings; graphic contraction mappings

JEL Classification: 47H10; 54H25

1. Introduction and Preliminaries

First, we present some definitions and basic notions of partial-metric, metric-like, b-metric, partial b-metric and b-metric-like spaces as the generalizations of standard metric spaces. After that, we give a process diagram, where arrows stand for generalization relationships.

Definition 1. *[1] Let X be a nonempty set. A mapping $p_{pm}: X \times X \to [0,+\infty)$ is said to be a p-metric if the following conditions hold for all $x,y,z \in X$:*

$(p_{pm}1)$ $x = y$ *if and only if* $p_{pm}(x,x) = p_{pm}(x,y) = p_{pm}(y,y)$;
$(p_{pm}2)$ $p_{pm}(x,x) \leq p_{pm}(x,y)$;
$(p_{pm}3)$ $p_{pm}(x,y) = p_{pm}(y,x)$;
$(p_{pm}4)$ $p_{pm}(x,y) \leq p_{pm}(x,z) + p_{pm}(z,y) - p_{pm}(z,z)$.

Then, the pair (X, p_{pm}) is called a partial metric space.

Definition 2. *[2] Let X be a nonempty set. A mapping $b_{ml}: X \times X \to [0,+\infty)$ is said to be metric-like if the following conditions hold for all $x,y,z \in X$:*

(b_l1) $b_{ml}(x,y) = 0$ *implies* $x = y$;
(b_l2) $b_{ml}(x,y) = b_{ml}(y,x)$;

$(b_l 3)$ $b_{ml}(x,z) \leq b_{ml}(x,y) + b_{ml}(y,z)$.
In this case, the pair (X, b_{ml}) *is called a metric-like space.*

Definition 3. *[3,4] Let* X *be a nonempty set and* $s \geq 1$ *a given real number. A mapping* $b : X \times X \to [0, +\infty)$ *is called a b-metric on the set* X *if the following conditions hold for all* $x, y, z \in X$:

$(b1)$ $b(x,y) = 0$ *if and only if* $x = y$;
$(b2)$ $b(x,y) = b(y,x)$;
$(b3)$ $b(x,z) \leq s[b(x,y) + b(y,z)]$.

In this case, the pair (X, b) *is called a b-metric space (with coefficient* $s \geq 1$*).*

Definition 4. *[5,6] Let* X *be a nonempty set and* $s \geq 1$. *A mapping* $b_{pb} : X \times X \to [0, +\infty)$ *is called a partial b-metric on the set* X *if the following conditions hold for all* $x, y, z \in X$:

$(b_{pb}1)$ $x = y$ *if and only if* $p_{pb}(x,x) = p_{pb}(x,y) = p_{pb}(y,y)$;
$(b_{pb}2)$ $b_{pb}(x,x) \leq b_{pb}(x,y)$;
$(b_{pb}3)$ $b_{pb}(x,y) = b_{pb}(y,x)$;
$(b_{pb}4)$ $b_{pb}(x,y) \leq s\left[b_{pb}(x,z) + b_{pb}(z,y)\right] - b_{pb}(z,z)$.

Then, the pair (X, b_{pb}) *is called a partial b-metric space.*

Definition 5. *[7] Let* X *be a nonempty set and* $s \geq 1$. *A mapping* $b_{bl} : X \times X \to [0, +\infty)$ *is called b-metric-like on the set* X *if the following conditions hold for all* $x, y, z \in X$:

$(b_{bl}1)$ $b_{bl}(x,y) = 0$ *implies* $x = y$;
$(b_{bl}2)$ $b_{bl}(x,y) = b_{bl}(y,x)$;
$(b_{bl}3)$ $b_{bl}(x,z) \leq s[b_{bl}(x,y) + b_{bl}(y,z)]$.

In this case, the pair (X, b_{bl}) *is called a b-metric-like space with coefficient* $s \geq 1$.

Now, we give the process diagram of the classes of generalized metric spaces that were introduced earlier:

Metric space	$\to$	Partial metric space	$\to$	Metric-like space
$\downarrow$		$\downarrow$		$\downarrow$
b-Metric space	$\to$	Partial b-metric space	$\to$	b-Metric-like space

For more details on other generalized metric spaces see [8–14].

The next proposition helps us to construct some more examples of b-metric (respectively partial b-metric, b-metric-like) spaces.

Proposition 1. *Let* (X, d) *(resp.* (X, p_{pm}), (X, b_{ml})*) be a metric (resp. partial metric, metric-like) space and* $D(x,y) = (d(x,y))^k$ *(resp.* $P_{pm}(x,y) = (p_{pm}(x,y))^k$, $B_{ml}(x,y) = (b_{ml}(x,y))^k$*), where* $k > 1$ *is a real number. Then* D *(resp.* P_{pm}, B_{pm}*) is b-metric (resp. partial b-metric, b-metric-like) with coefficient* $s = 2^{k-1}$.

Proof. The proof follows from the fact that

$$u^k + v^k \leq (u+v)^k \leq (a+b)^k \leq 2^{k-1}\left(a^k + b^k\right),$$

for all nonnegative real numbers a, b, u, v with $u + v \leq a + b$. □

It is clear that each metric-like space, i.e., each partial b-metric space, is a b-metric-like space, while the converse is not true. For more such examples and details see [1,2,5–7,15–27]. Moreover, for various metrics in the context of the complex domain see [28,29].

The definitions of convergent and Cauchy sequences are formally the same in partial metric, metric-like, partial b-metric and b-metric like spaces. Therefore, we give only the definition of convergence and Cauchyness of the sequences in b-metric-like space. Moreover, these two notions are formally the same in metric and b-metric spaces.

Definition 6. *[7] Let $\{x_n\}$ be a sequence in a b-metric-like space (X, b_{bl}) with coefficient s.*

(i) The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n\to\infty} b_{bl}(x_n, x) = b_{bl}(x, x)$;

(ii) The sequence $\{x_n\}$ is said to be b_{bl}-Cauchy in (X, b_{bl}) if $\lim_{n,m\to\infty} b_{bl}(x_n, x_m)$ exists and is finite;

(iii) One says that a b-metric-like space (X, b_{bl}) is b_{bl}-complete if for every b_{bl}-Cauchy sequence $\{x_n\}$ in X there exists an $x \in X$, such that $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = b_{bl}(x, x) = \lim_{n\to\infty} b_{bl}(x_n, x)$.

Remark 1. *In a b-metric-like space the limit of a sequence need not be unique and a convergent sequence need not be a b_{bl}-Cauchy sequence (see Example 7 in [18]). However, if the sequence $\{x_n\}$ is b_{bl}-Cauchy with $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = 0$ in the b_{bl}-complete b-metric-like space (X, b_{bl}) with coefficient $s \geq 1$, then the limit of such a sequence is unique. Indeed, in such a case if $x_n \to x$ $(b_{bl}(x_n, x) \to b_{bl}(x, x))$ as $n \to \infty$ we get that $b_{bl}(x, x) = 0$. Now, if $x_n \to x$ and $x_n \to y$ where $x \neq y$, we obtain that:*

$$\frac{1}{s} b_{bl}(x, y) \leq b_{bl}(x, x_n) + b_{bl}(x_n, y) \to b_{bl}(x, x) + b_{bl}(y, y) = 0 + 0 = 0. \tag{1}$$

From $(b_{bl}1)$ it follows that $x = y$, which is a contradiction. The same is true as well for partial metric, metric like and partial b-metric spaces.

The next definition and the corresponding proposition are important in the context of fixed point theory.

Definition 7. *[30] The self-mappings $f, g : X \to X$ are weakly compatible if $f(g(x)) = g(f(x))$, whenever $f(x) = g(x)$.*

Proposition 2. *[30] Let T and S be weakly compatible self-maps of a nonempty set X. If they have a unique point of coincidence $w = f(u) = g(u)$, then w is the unique common fixed point of f and g.*

In this paper we shall use the following result to prove that certain Picard sequences are Cauchy. The proof is completely identical with the corresponding in [31] (see also [25]).

Lemma 1. *Let $\{x_n\}$ be a sequence in a b-metric-like space (X, b_{bl}) with coefficient $s \geq 1$ such that*

$$b_{bl}(x_n, x_{n+1}) \leq \lambda b_{bl}(x_{n-1}, x_n) \tag{2}$$

for some $\lambda, 0 \leq \lambda < \frac{1}{s}$, and each $n = 1, 2, ...$Then $\{x_n\}$ is a b_{bl}-Cauchy sequence in (X, b_{bl}) such that $\lim_{n,m\to\infty} b_{bl}(x_n, x_m) = 0$.

Remark 2. *It is worth noting that the previous lemma holds in the context of b-metric-like spaces for each $\lambda \in [0, 1)$. For more details see [6,32].*

2. Main Results

In line with Jachymski [33], let (X, b_{bl}) be a b-metric-like space and $\mathcal{D}$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \mathcal{D}$. We also assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning the distance between its vertices to each edge (see [33]).

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus, we have

$$E\left(G^{-1}\right) = \{(x, y) \in X \times X : (y, x) \in E(G)\}. \tag{3}$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric under the convention

$$E\left(\widetilde{G}\right) = E(G) \cup E\left(G^{-1}\right). \tag{4}$$

If x and y are vertices in a graph G, then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected.

Recently, some results have appeared providing sufficient conditions for a self mapping of X to be a Picard operator when (X, d) is endowed with a graph. The first result in this direction was given by Jachymski [33]. Moreover, see [34–36].

Definition 8. *[33] We say that a mapping $f : X \to X$ is a Banach G-contraction or simply a G-contraction if f preserves edges of G, i.e.,*

$$\text{for all } x, y \in X : (x, y) \in E(G) \text{ implies } (f(x), f(y)) \in E(G) \tag{5}$$

and f decreases the weights of edges of G as for all $x, y \in X$, there exists $\lambda \in (0, 1)$, such that

$$(x, y) \in E(G) \text{ implies } d(f(x), f(y)) \leq \lambda d(x, y). \tag{6}$$

Definition 9. *[37] A mapping $g : X \to X$ is called orbitally continuous, if given $x \in X$ and any sequence $\{k_n\}$ of positive integers,*

$$g^{k_n}(x) \to y \text{ as } n \to \infty \text{ implies } g\left(g^{k_n}(x)\right) \to g(y) \text{ as } n \to \infty. \tag{7}$$

Definition 10. *[33] A mapping $g : X \to X$ is called G-continuous, if for any given $x \in X$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with the properties that for all $n \in \mathbb{N}$ the pair $(x_n, x_{n+1}) \in E(G)$ and that $x_n \to x$ as $n \to \infty$ it follows that $g(x_n) \to g(x)$.*

Definition 11. *[33] A mapping $g : X \to X$ is called orbitally G-continuous, if given $x, y \in X$ and any sequence $\{k_n\}$ of positive integers for all $n \in \mathbb{N}$,*

$$g^{k_n}x \to y \text{ and } \left(g^{k_n}(x), g^{k_n+1}(x)\right) \in E(G) \text{ implies } g\left(g^{k_n}(x)\right) \to g(y) \text{ as } n \to \infty. \tag{8}$$

In this section, we consider self-mappings $f, g : X \to X$ with $f(X) \subset g(X)$. Let $x_0 \in X$ be an arbitrary point, then there exists $x_1 \in X$ such that $z_0 = f(x_0) = g(x_1)$. By repeating this step we can build a sequence $\{z_n\}$ such that $z_n = f(x_n) = g(x_{n+1})$ and the following property:

The property $G_{f,g(x_n)}$. If $\{g(x_n)\}_{n\in\mathbb{N}}$ is a sequence in X such that $(g(x_n), g(x_{n+1})) \in E(G)$ for all $n \geq 1$ and $g(x_n) \to x$, then there is a subsequence $\{g(x_{n_i})\}_{i\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $(g(x_{n_i}), x) \in E(G)$ for all $i \geq 1$. Note that the property $G_{f,g(x_n)}$ depends only on the pair of mappings f and g, and does not depend on the sequence $\{x_n\}$. Here, we use notation G_{gf} in the following sense: $x \in X$ belongs to G_{gf} if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_0 = x$, $f(x_{n-1}) = g(x_n)$ for $n \in \mathbb{N}$, and $(g(x_n), g(x_m)) \in E(G)$ for all $m, n \in \mathbb{N}$.

Now, we present the first result of this section.

Theorem 1. *(Hardy-Rogers) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and which satisfy*

$$\begin{aligned} s^q b_{bl}(f(x), f(y)) &\leq c_1 b_{bl}(g(x), g(y)) + c_2 b_{bl}(g(x), f(x)) + c_3 b_{bl}(g(y), f(y)) \\ &\quad + c_4 b_{bl}(g(x), f(y)) + c_5 b_{bl}(g(y), f(x)), \end{aligned} \tag{9}$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$ where $q \geq 2, c_i \geq 0, i = 1, \ldots, 5$ and either

$$c_1 + c_2 + c_3 + 2c_4 + 2c_5 < \frac{1}{s} \tag{10}$$

or

$$c_1 + 2c_2 + 2c_3 + c_4 + c_5 < \frac{1}{s}. \tag{11}$$

Suppose that $f(X) \subset g(X)$ and at least one of $f(X), g(X)$ is b_{bl}-complete subspace of (X, b_{bl}). Then:

(i) If the pair (f, g) has property $G_{f,g(x_n)}$ and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Proof. (i) Assume that $G_{gf} \neq \emptyset$, there exists $x_0 \in G_{gf}$. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $f(x_0) = g(x_1)$, again we can find $x_2 \in X$ such that $f(x_1) = g(x_2)$. Repeating this step, we can build a sequence $z_n = f(x_n) = g(x_{n+1})$ such that $(z_n, z_m) \in E(G)$. If $z_k = z_{k+1}$ for some $k \in \mathbb{N}$, then $f(x_{k+1}) = g(x_{k+1})$ is a point of coincidence of f and g. Therefore, let $z_n \neq z_{n+1}$ for all $n \in \mathbb{N}$. By Condition (9), we can get that

$$\begin{aligned} b_{bl}(z_n, z_{n+1}) &\leq s^q b_{bl}(z_n, z_{n+1}) = s^q b_{bl}(f(x_n), f(x_{n+1})) \\ &\leq c_1 b_{bl}(g(x_n), g(x_{n+1})) + c_2 b_{bl}(g(x_n), f(x_n)) + c_3 b_{bl}(g(x_{n+1}), f(x_{n+1})) \\ &\quad + c_4 b_{bl}(g(x_n), f(x_{n+1})) + c_5 b_{bl}(g(x_{n+1}), f(x_n)). \end{aligned} \tag{12}$$

Since $z_n = f(x_n) = g(x_{n+1})$ then Condition (12) becomes

$$\begin{aligned} b_{bl}(z_n, z_{n+1}) &\leq c_1 b_{bl}(z_{n-1}, z_n) + c_2 b_{bl}(z_{n-1}, z_n) + c_3 b_{bl}(z_n, z_{n+1}) \\ &\quad + c_4 b_{bl}(z_{n-1}, z_{n+1}) + c_5 b_{bl}(z_n, z_n) \\ &\leq c_1 b_{bl}(z_{n-1}, z_n) + c_2 b_{bl}(z_{n-1}, z_n) + c_3 b_{bl}(z_n, z_{n+1}) + s c_4 b_{bl}(z_{n-1}, z_n) \\ &\quad + s c_4 b_{bl}(z_n, z_{n+1}) + 2 s c_5 b_{bl}(z_{n-1}, z_n), \end{aligned} \tag{13}$$

or equivalently:

$$b_{bl}(z_n, z_{n+1}) \leq \lambda b_{bl}(z_{n-1}, z_n), \tag{14}$$

where $\lambda = \frac{c_1 + c_2 + s c_4 + 2 s c_5}{1 - c_3 - s c_4}$. Since, $c_1 + c_2 + c_3 + s c_4 + 2 s c_5 \leq s c_1 + s c_2 + s c_3 + 2 s c_4 + 2 s c_5 < 1$, it follows that $\lambda < 1$. Therefore, by Remark 2 of Lemma 1, the sequence $z_n = f(x_n) = g(x_{n+1})$ is a b_{bl}-Cauchy sequence. The b_{bl}-completeness of $f(X)$ leads to $u \in f(X) \subset g(X)$ such that $z_n \to u = g(v)$ for some $v \in X$. As $z_0 \in G_{gf}$, this implies that $(z_n, z_m) \in E(G)$ for $n, m = 1, 2, \ldots$ and so $(z_n, z_{n+1}) \in E(G)$.

By property $G_{f,g(x_n)}$, there is a subsequence $\{z_{n_i}\}_{i\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ such that $(z_{n_i},u)\in E(G)$. Applying $(b_{bl}3)$, we get

$$\begin{aligned}
b_{bl}(f(v),g(v)) &\leq sb_{bl}(f(v),f(x_{n_i}))+sb_{bl}(f(x_{n_i}),g(v)) \\
&\leq s^q b_{bl}(f(v),f(x_{n_i}))+sb_{bl}(f(x_{n_i}),g(v)) \\
&\leq c_1 b_{bl}(g(v),g(x_{n_i}))+c_2 b_{bl}(g(v),f(v))+c_3 b_{bl}(g(x_{n_i}),f(x_{n_i})) \\
&\leq +c_4 b_{bl}(g(v),f(x_{n_i}))+c_5 b_{bl}(g(x_{n_i}),f(v))+sb_{bl}(f(x_{n_i},g(v))) \\
&= c_1 b_{bl}(g(v),z_{n_i-1})+c_2 b_{bl}(g(v),f(v))+c_3 b_{bl}(z_{n_i-1},z_{n_i}) \\
&\quad +c_4 b_{bl}(g(v),z_{n_i})+c_5 b_{bl}(z_{n_i-1},f(v))+sb_{bl}(z_{n_i},g(v)).
\end{aligned} \tag{15}$$

Since $b_{bl}(z_{n_i-1},f(v))\leq sb_{bl}(z_{n_i-1},g(v))+sb_{bl}(g(v),f(v))$, Condition (15) becomes

$$\begin{aligned}
&(1-c_2-c_5 s)\,b_{bl}(f(v),g(v)) \\
\leq\ & c_1 b_{bl}(g(v),z_{n_i-1})+c_3 b_{bl}(z_{n_i-1},z_{n_i})+c_4 b_{bl}(g(v),z_{n_i}) \\
& +c_5 s b_{bl}(z_{n_i-1},g(v))+sb_{bl}(z_{n_i},g(v)).
\end{aligned} \tag{16}$$

Taking the limit in Condition (16) as $i\to\infty$ we obtain that $b_{bl}(f(v),g(v))=0$, because $c_2+c_5 s\leq c_1 s+c_2 s+c_3 s+2c_4 s+2c_5 s<1$. That is, $f(v)=g(v)=u$ is a point of coincidence for the mappings f and g, i.e., (i) is proved in the case if $f(X)$ is b_{bl}-complete. The proof for the case if $g(X)$ is b_{bl}-complete is similar.

(ii) Assume that x and y are two different points of coincidence of f and g with $(x,y)\in E(G)$. This means that there are different points x_1 and y_1 from X such that: $f(x_1)=g(x_1)=x$ and $f(y_1)=g(y_1)=y$. Now, according to Condition (9) we get

$$\begin{aligned}
sb_{bl}(x,y) &\leq s^q b_{bl}(x,y)=s^q b_{bl}(f(x_1),f(y_1)) \\
&\leq c_1 b_{bl}(g(x_1),g(y_1))+c_2 b_{bl}(g(x_1),f(y_1))+c_3 b_{bl}(g(y_1),f(y_1)) \\
&\quad +c_4 b_{bl}(g(x_1),f(y_1))+c_5 b_{bl}(g(y_1),f(x_1)) \\
&= c_1 b_{bl}(x,y)+c_2 b_{bl}(x,y)+c_3 b_{bl}(y,y) \\
&\quad +c_4 b_{bl}(x,y)+c_5 b_{bl}(y,x) \\
&\leq (c_1+c_2+2c_3 s+c_4+c_5)\,b_{bl}(y,x) \\
&\leq (c_1 s+2c_2 s+2c_3 s+c_4 s+c_5 s)\,b_{bl}(y,x)<b_{bl}(y,x).
\end{aligned} \tag{17}$$

Hence, if $x\neq y$ we get a contradiction.

If f and g are weakly compatible, then by Proposition 2 f and g have a unique common fixed point. □

Example 1. *Let $X=[0,+\infty)$ and $f,g:X\to X$ be the mappings such that*

$$f(x)=e^x-1 \quad \text{and} \quad g(x)=e^{4x}-1.$$

Consider b-metric-like space (X,b_{bl}) under the distance $b_{bl}(x,y)=(x+y)^2$ with coefficient $s=2$, and the graph $G=(V,E)$ with $V=X$ and $E=\{(x,x):x\in X\}\cup\{(0,x):x\in X\}$. Assume that $c_1=\frac{1}{4}$ and $c_2=c_3=c_4=c_5=\frac{1}{25}$ for which Inequalities (10) and (11) hold. Note that $(g(x),g(y))\in E$ if and only if $x=y,x\geq 0$ or $x=0,y>0$ or $y=0,x>0$. For $q=2$ let us check whether Condition (9) holds in these cases.

Case 1: $x = y, x \geq 0$;

$$\begin{aligned} & c_1 b_{bl}(g(x), g(x)) + c_2 b_{bl}(g(x), f(x)) + c_3 b_{bl}(g(x), f(x)) + c_4 b_{bl}(g(x), f(x)) + c_5 b_{bl}(g(x), f(x)) \\ = \; & c_1 \left(e^{4x} - 1 + e^{4x} - 1\right)^2 + (c_2 + c_3 + c_4 + c_5) \left(e^{4x} - 1 + e^x - 1\right)^2 \\ = \; & 4c_1 \left(e^x - 1\right)^2 \left(e^{3x} + e^{2x} + e^x + 1\right)^2 + (c_2 + c_3 + c_4 + c_5) \left(e^x - 1\right)^2 \left(e^{3x} + e^{2x} + e^x + 2\right)^2 \\ \geq \; & 4c_1 \left(e^x - 1\right)^2 4^2 + (c_2 + c_3 + c_4 + c_5) \left(e^x - 1\right)^2 5^2 = \left(\tfrac{1}{4} \cdot 64 + \tfrac{4}{25} \cdot 25\right) \left(e^x - 1\right)^2 \\ > \; & 4 \left(e^x - 1 + e^x - 1\right)^2 = s^q b_{bl}(f(x), f(x)). \end{aligned}$$

Case 2: $x = 0, y > 0$ *(similarly for* $y = 0, x > 0$*)*;

$$\begin{aligned} & c_1 b_{bl}(g(0), g(y)) + c_2 b_{bl}(g(0), f(0)) + c_3 b_{bl}(g(y), f(y)) + c_4 b_{bl}(g(0), f(y)) + c_5 b_{bl}(g(y), f(0)) \\ = \; & c_1 \left(e^{4y} - 1\right)^2 + c_2 (0 + 0)^2 + c_3 \left(e^{4y} - 1 + e^y - 1\right)^2 + c_4 \left(e^y - 1\right)^2 + c_5 \left(e^{4y} - 1\right)^2 \\ = \; & (c_1 + c_5) \left(e^y - 1\right)^2 \left(e^{3y} + e^{2y} + e^y + 1\right)^2 + c_3 \left(e^y - 1\right)^2 \left(e^{3y} + e^{2y} + e^y + 2\right)^2 + c_4 \left(e^y - 1\right)^2 \\ > \; & (c_1 + c_5) \left(e^y - 1\right)^2 4^2 + c_3 \left(e^y - 1\right)^2 5^2 + c_4 \left(e^y - 1\right)^2 = \left(\tfrac{29}{100} \cdot 16 + \tfrac{1}{25} \cdot 25 + \tfrac{1}{25}\right) \left(e^y - 1\right)^2 \\ > \; & 4 \left(e^y - 1\right)^2 = s^q b_{bl}(f(0), f(y)). \end{aligned}$$

Hence, f and g satisfy Condition (9) for all $x, y \in X$ such that $(g(x), g(y)) \in E$.

Moreover, there is $x_1 = \frac{x_0}{4}$ such that $g(x_1) = f(x_0)$, $x_2 = \frac{x_0}{4^2}$ such that $g(x_2) = f(x_1)$, and so on. In this way, we can built the sequence $x_n = \frac{x_0}{4^n}, n \in \mathbb{N}$ such that $g(x_n) = f(x_{n-1})$. For $x_0 \neq 0$ it is clear that $(g(x_n), g(x_m)) \notin E$. For $x_0 = 0$, $x_n = 0, n \in \mathbb{N}$ is obtained. Thus, the constant sequence $x_n = 0$ is only convergent sequence such that $(g(x_n), g(x_m)) = (0, 0) \in E$, and for each subsequence $(g(x_{n_i}))_{i \in \mathbb{N}}$ of $(g(x_n))_{n \in \mathbb{N}}$ holds $(g(x_{n_i}), 0) = (0, 0) \in E$. This means that $x_0 \in G_{gf} \neq \emptyset$ and the pair (f, g) possesses the property $G_{f, g(x_n)}$.

It is obvious that $f(X) \subset g(X)$ and $g(X) = X$ is b_{bl}-complete. Since the mappings f and g are weakly compatible at $x = 0$ ($f(0) = g(0)$ implies $g(f(0)) = f(g(0))$), all conditions of Theorem 1 are satisfied. So, 0 is the unique common fixed point of mappings f and g in X.

Example 2. *Now consider the same b-metric-like space (X, b_{bl}) endowed with the graph G as in Example 1, and the mappings $f, g : X \to X$ such that*

$$f(x) = \begin{cases} e^x - 1\,, & x \neq 0 \\ \quad 1 \,, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} e^{4x} - 1\,, & x \neq 0 \\ \quad 2 \,, & x = 0 \end{cases}.$$

In this case we have $G_{gf} = \emptyset$. Namely, for $x_0 = 0$, $x_n = \frac{1}{4^n} \ln 2, n \in \mathbb{N}$ is now obtained, and $(g(x_n), g(x_m)) \notin E$. Hence, the conditions of Theorem 1 are not satisfied. Moreover, we can easily see that the mappings f and g have no coincidence point nor common fixed points.

As corollaries of our Theorem 1, we obtain the next results in the context of *b*-metric-like spaces endowed with a graph:

Corollary 1. *(Jungck) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl} \left(f\left(x\right), f\left(y\right)\right) \leq c_1 b_{bl} \left(g\left(x\right), g\left(y\right)\right) \tag{18}$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$ when $c_1 < \frac{1}{s}$. Suppose that $f(X) \subset g(X)$ and at least one of $f(X), g(X)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f, g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 2. *(Kannan) Let $f,g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_2 b_{bl}\left(g\left(x\right), f\left(x\right)\right) + c_3 b_{bl}\left(g\left(y\right), f\left(y\right)\right) \tag{19}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_2 + c_3 < \frac{1}{2s}. \tag{20}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E\left(G\right)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 3. *(Chatterjea) Let $f,g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_4 b_{bl}\left(g\left(x\right), f\left(y\right)\right) + c_5 b_{bl}\left(g\left(y\right), f\left(x\right)\right), \tag{21}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_4 + c_5 < \frac{1}{2s}. \tag{22}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E\left(G\right)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Corollary 4. *(Reich) Let $f,g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}\left(f\left(x\right), f\left(y\right)\right) \leq c_1 b_{bl}\left(g\left(x\right), g\left(y\right)\right) + c_2 b_{bl}\left(g\left(x\right), f\left(x\right)\right) + c_3 b_{bl}\left(g\left(y\right), f\left(y\right)\right) \tag{23}$$

for all $x, y \in X$ with $\left(g\left(x\right), g\left(y\right)\right) \in E\left(G\right)$ when

$$c_1 + 2c_2 + 2c_3 < \frac{1}{s} \tag{24}$$

Suppose that $f\left(X\right) \subset g\left(X\right)$ and at least one of $f\left(X\right), g\left(X\right)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E\left(G\right)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Now, we announce our last result in this section in the context of b-metric-like spaces endowed with the graph. The proof is similar enough with the corresponding proof of Theorem 1 and therefore we omit it.

Theorem 2. *(Das-Naik-Ćirić) Let $f, g : X \to X$ be self-mappings defined on a b-metric-like space (X, b_{bl}) (with coefficient $s \geq 1$) endowed with a graph G, and satisfy*

$$s^q b_{bl}(f(x), f(y)) \leq \lambda \max\{b_{bl}(g(x), g(y)), b_{bl}(g(x), f(x)), b_{bl}(g(y), f(y)), b_{bl}(g(x), f(y)), b_{bl}(g(y), f(x))\} \quad (25)$$

for all $x, y \in X$ with $(g(x), g(y)) \in E(G)$ when $\lambda \in [0, \frac{1}{s})$. Suppose that $f(X) \subset g(X)$ and at least one of $f(X), g(X)$ is a b_{bl}-complete subspace of (X, b_{bl}). Then

(i) If the property $G_{f,g(x_n)}$ is satisfied and $G_{gf} \neq \emptyset$, then f and g have a point of coincidence in X.

(ii) If x and y in X are points of coincidence of f and g such that $(x, y) \in E(G)$, then $x = y$. Hence, points of coincidence of f and g are unique in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.

Author Contributions: Conceptualization, T.D. and S.R.; Data curation, M.P.; Formal analysis, N.N. and T.D.; Investigation, N.N.; Methodology, M.D.l.S., M.P., N.N. and T.D.; Project administration, M.D.l.S.; Supervision, M.D.l.S. and S.R.

Funding: This research was funded by the Basque Government, grant number IT1207-19; the Serbian Ministry of Science and Technology, grant number OI 174010 and the Ministry of Education and Science and Technological Development of the Republic of Serbia, Grant number -174002 . The APC was funded by the Basque Government through grant IT1207-19.

Acknowledgments: The first author thanks the Basque Government for its support of this work through Grant IT1207-19, the second author is supported by the Serbian Ministry of Science and Technology OI 174010, the third author wish to thank the projects MNTRRS-174009, the fourth author is grateful for the financial support from the Ministry of Education and Science and Technological Development of the Republic of Serbia-174002.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Matthews, S.G. Partial metric topology. In Proceedings of the 8th Summer Conference on General Topology and Applications, Queens College, New York, NY, USA, 18–20 June 1992; Annals of the New York Academy of Sciences: New York, NY, USA, 1994; pp. 183–197.
2. Harandi, A.A. Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.* **2012**, *2012*, 1–10.
3. Bakhtin, I.A. The contraction mapping principle in quasi-metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
4. Czerwik, S. Contraction mappings in *b*-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
5. Shukla, S. Partial *b*-metric spaces and fixed point theorems. *Mediterranean J. Math.* **2014**, *11*, 703–711. [CrossRef]
6. Vujaković, J.; Aydi, H.; Radenović, S.; Mukheimer, A. Some remarks and new results in ordered partial *b*-metric spaces. *Mathematics* **2019**, *7*, 334. [CrossRef]
7. Alghmandi, M.A.; Hussain, N.; Salimi, P. Fixed point and coupled fixed point theorems on *b*-metric-like spaces. *J. Inequal. Appl.* **2013**, *402*, 1–25.
8. Agarwal, R.P.; Karapinar, E.; O'Regan, D.; Roldan-Lopez-de-Hierro, A.F. *Fixed Point Theory in Metric Type Spaces*; Springer International Publishing: Cham, Switzerland, 2015. [CrossRef]
9. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
10. Gaba, Y.U.; Agyingi, C.A.; Leko, D.J. Chatterjea type fixed point in partial *b*-metric spaces. *arXiv* 2019, arXiv:1902.03108v1.
11. George, R.; Nabwey, H.A.; Ramaswamy, R.; Radenović, S. Some generalized contraction classes and common fixed points in *b*-metric spaces endowed with a graph. *Mathematics* **2019**, *7*, 754. [CrossRef]
12. George, R.; Tamrakar, E.; Vujaković, J.; Pathak, H.K.; Velusamy, S. (C, ψ^*, G) class of contractions and fixed points in a metric space endowed with a graph. *Mathematics* **2019**, *7*, 482. [CrossRef]
13. Kirk, W.A.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer International Publishing: Cham, Switzerland, 2014. [CrossRef]

14. Rhoades, B.E. Comparison of Various Definitions of Contractive Mappings. *Trans. Am. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
15. Alsulami, H.; Gulyaz, S.; Karapinar, E.; Erha, I.M. An Ulam stability result on quasi-b-metric like spaces. *Open Math.* **2016**, *14*, 1–15. [CrossRef]
16. Aydi, H.; Felhi, A.; Sahmim, S. On common fixed points for (α, ψ)-contractions and generalized cyclic contractions in b-metric-like spaces and consequences. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2492–2510. [CrossRef]
17. Aydi, H.; Felhi, A.; Sahmim, S. Common fixed points via implicit contractions on b-metric-like spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1524–1537. [CrossRef]
18. Hammad, H.A.; De la Sen, M. Generalized contractive mappings and related results in b-metric-like spaces with an application. *Symmetry* **2019**, *11*, 667. [CrossRef]
19. Hussain, N.; Roshan, J.R.; Parvaneh, V.; Kadelburg, Z. Fixed points of contractive mappings in b-metric-like spaces. *Sci. World J.* **2014**, 471827. [CrossRef] [PubMed]
20. Mukheimer, A. $\alpha - \psi - \varphi$ contractive mappings in ordered partial b-metric spaces. *J. Nonlinear Sci. Appl.* **2014**, *7*, 168–179. [CrossRef]
21. Mustafa, Z.; Jaradat, M.M.M.; Jaradat, H.M. Some common fixed point results of graph on b-metric space. *J. Nonlinear Sci. Appl.* **2016**, *9*, 4838–4851. [CrossRef]
22. Nashine, H.K.; Kadelburg, Z. Existence of solutions of Cantilever Beam Problem via α-β-FG-contractions in b-metric-like spaces. *Filomat* **2017**, *31*, 3057–3074. [CrossRef]
23. Padhan, S.K.; Jagannadha Rao, G.V.V.; Nashine, H.K.; Agarwal, R.P. Some fixed point results for (β-ψ_1-ψ_2)-contractive conditions in ordered b-metric-like spaces. *Filomat* **2017**, *31*, 4587–4612. [CrossRef]
24. Shukla, S.; Radenović, S.; Rajić, V. Ć. Some common fixed point theorems in 0-σ-complete metric-like spaces. *Vietnam J. Math.* **2013**, *41*, 341–352. [CrossRef]
25. Singh, S.L.; Czerwik, S.; Krol, K.; Singh, A. Coincidences and fixed points of hybrid contractions. *Tamsui Oxf. J. Math. Sci.* **2008**, *24*, 401–416.
26. Zoto, K.; Radenović, S.; Ansari, A.H. On some fixed point results for (s, p, α)-contractive mappings in b-metric-like spaces and applications to integral equations. *Open Math.* **2018**, *16*, 235–249. [CrossRef]
27. Zoto, K.; Rhoades, B.E.; Radenović, S. Some generalizations for (α-ψ, ϕ)-contractions in b-metric-like spaces and an application. *Fixed Point Theory Appl.* **2017**, *26*. [CrossRef]
28. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer Nature: Cham, Switzerland, 2019. [CrossRef]
29. Todorčević, V. Subharmonic behavior and quasiconformal mappings. *Anal. Math. Phys.* **2019**, *9*, 1211–1225. [CrossRef]
30. Abbas, M.; Jungck, G. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. *J. Math. Anal. Appl.* **2008**, *341*, 416–420. [CrossRef]
31. Jovanović, M.; Kadelburg, Z.; Radenović, S. Common fixed point results in metric-type spaces. *Fixed Point Theory Appl.* **2010**, 978121. [CrossRef]
32. Aleksić, S.; Mitrović, Z.D.; Radenović, S. Picard sequences in b-metric spaces. *Fixed Point Theory* 2019, accepted.
33. Jachymski, J. The contraction principle for mappings on a metric space with a graph. *Proc. Am. Math. Soc.* **2008**, *1*, 1359–1373. [CrossRef]
34. Beg, I.; But, A.R.; Radojević, S. The contraction principle for set valued mappings on a metric space with a graph. *Comput. Math. Appl.* **2010**, *60*, 1214–1219. [CrossRef]
35. Bojor, F. Fixed point of ϕ-contraction in metric spaces endowed with a graph. *Ann. Univ. Craiova Ser. Math. Inform.* **2010**, *37*, 85–92.
36. Bojor, F. Fixed point theorems for Reich type contractions on metric spaces with graph. *Nonlinear Anal.* **2012**, *75*, 3895–3901. [CrossRef]
37. Ćirić, Lj. *Some Recent Results in Metrical Fixed Point Theory*; University of Belgrade: Belgrade, Serbia, 2003.