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Representations of general solutions to some classes of nonlinear difference equations

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Abstract

Representations of general solutions to three related classes of nonlinear difference equations in terms of specially chosen solutions to linear difference equations with constant coefficients are given. Our results considerably extend some results in the literature and give theoretical explanations for them.

MSC: Primary 39A10; secondary 39A06; 39A45

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1 Introduction

Throughout the paper $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of natural, nonnegative, integer, real and complex numbers, respectively, whereas for $k, l \in \mathbb{Z}$, the notation $j = \overline{k, l}$ denotes the set of all integers j such that $k \leq j \leq l$.

No doubt that solvability of difference equations, as one of the basic and oldest topics in the theory of difference equations, is if not the most interesting then certainly one of such topics. First results on the topic were obtained by de Moivre in a series of papers and books (see, e.g., [1, 2]). His methods and ideas were later developed by Euler [3]. Further important results were obtained in the second half of the eighteenth century by several mathematicians, predominately by Lagrange and Laplace. What was known about difference equations up to the end of the nineteenth century was more or less summarized in books [4, 5]. Bearing in mind that many new general classes of solvable difference equations have not been found during the nineteenth century, researchers interested in difference equations turned to some other topics such as finding approximate solutions to equations, qualitative theory of equations, etc. The turn can be easily noticed in books on difference equations in the twentieth century (see [6–13]).

In the last few decades many authors used computers when dealing with difference equations. Some of the authors use computers for making conjectures on long-term behavior of their solutions based on numerical calculations, but some use them for getting closed-form formulas for their solutions in the "experimental" as well as direct way by symbolic algebraic computations. Getting formulas for the solutions of the equations without knowing some theory leads frequently to some problems, especially related to the novelty of the formulas. Some of our papers were devoted to such problems (see, e.g., [14–20]), where we gave theoretical explanations of some of the formulas obtained in that way.

Let us also mention that a constant interest in the topic has been kept in popular journals for a wide mathematical audience and in problem books on "elementary" mathematics (see, e.g., [21–25]).

Our renewed interest in solvability of difference equations started in 2004, when Stević gave a theoretical explanation for the solvability of the following nonlinear difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n}, \quad n \in \mathbb{N}_0, \tag{1}$$

which was, in fact, the first case of giving theoretical explanations for some formulas, for which no theory or explanations how they had been obtained were given.

Modifications of the method used in solving equation (1) have been later used many times (see, e.g., [16, 17, 26–28]). The main idea is to transform a nonlinear equation to a linear one. If the transformed equations are with constant coefficients, then they are solvable (see [4–7, 10, 11, 13]), which implies the solvability of the original equations. Motivated by some investigations of symmetric and closely related systems of difference equations, for example, those in [29–32], we have also started studying solvability of such systems of difference equations related to the solvable difference equations mentioned above (see, e.g., [16, 33–35]). For example, the corresponding close-to-symmetric system of difference equations to equation (1) is the following one:

$$x_{n+1} = \frac{x_{n-1}}{a_n + b_n x_{n-1}y_n}, \qquad y_{n+1} = \frac{y_{n-1}}{c_n + d_n y_{n-1}x_n}, \quad n \in \mathbb{N}_0,$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, and $(d_n)_{n\in\mathbb{N}_0}$ are given sequences of numbers. The idea for studying such systems is similar, namely, to transform the (nonlinear) systems to linear ones. If they are with constant coefficients then they are solvable (see [4–7, 10, 11, 13]).

Solvable difference equations are important also because of their applicability. Various applications of solvable difference equations and systems can be found, for example, in [7–9, 11, 13, 24, 25, 36–44].

For some other related methods for dealing with difference equations and systems, let us mention studying and applying their invariants (see [45–48]), investigating product-type difference equations and systems (see [49–52]), and some other results, e.g., in [53–57].

There has been also a recent habit that some authors represent general solutions to some solvable difference equations and systems of difference equations in terms of the Fibonacci sequence, and as a rule without any explanation how the representations are obtained, and only using the mathematical induction in proving them. Why the Fibonacci sequence was chosen for the representations has not been explained. However, quite frequently such representations are known or easily follow from known formulas which can be easily found in the literature. For some comments on several papers of this kind, see, e.g., [14, 18, 20], where we presented representations of solutions to some general difference equations and systems which include some very special cases of the equations and systems from the literature.

Elabbasy and Elsayed [58] were among those who presented solutions to several difference equations in terms of the Fibonacci sequence. The following equation is one of the presented therein:

$$x_n = \frac{x_{n-2}x_{n-1}}{x_{n-2} + x_{n-1}}, \quad n \geq 2, \tag{2}$$

for which it was shown that its general solution is given by the following formula:

$$x_n = \frac{x_0 x_1}{x_1 f_{n-1} + x_0 f_n}, \tag{3}$$

for $n \in \mathbb{N}_0$, where $(f_n)_{n\in\mathbb{N}_0}$ is the Fibonacci sequence. Recall that the sequence is the solution to the following difference equation:

$$f_{n+1} = f_n + f_{n-1}, \quad n \in \mathbb{N}, \tag{4}$$

with the initial conditions $f_0 = 0$ and $f_1 = 1$, and that it can be naturally prolonged for negative values of indices by using the following consequence of recurrent relation (4):

$$f_{n-1} = f_{n+1} - f_n, \quad n \in \mathbb{Z} \setminus \mathbb{N}.$$

Some information on the sequence can be found, for example, in [24, 39, 59, 60]. Equation (2) is well-known, and representation (3) easily follows from a known one. A method for solving equation (2) can be found, for example, in [22].

Here we explain how a representation formula for solutions to a general equation, which includes that in (3), can be obtained in an elegant way. Then we show that the choice of the Fibonacci sequence is, in fact, an artificial one, and that many other representations exist, but in terms of some other sequences, which are solutions to some linear difference equations.

In [58] the authors also considered the following two difference equations of the third order:

$$x_n = \frac{x_{n-2}x_{n-3}}{x_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{5}$$

and

$$x_n = \frac{x_{n-1}x_{n-3}}{x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}_0, \tag{6}$$

where it was shown that the general solution to equation (5) has the following representation:

$$x_n = \frac{x_{-1}x_{-2}x_{-3}}{x_{-2}x_{-3}s_{n+1} + x_{-1}x_{-3}s_{n+2} + x_{-1}x_{-2}s_n}, \quad n \geq -3, \tag{7}$$

where $(s_n)_{n\geq -3}$ is the solution to the following difference equation:

$$s_n = s_{n-2} + s_{n-3}, \quad n \in \mathbb{N}_0, \tag{8}$$

such that

$$s_{-2} = s_{-1} = 0, \qquad s_0 = 1, \tag{9}$$

and that the general solution to equation (6) has the following representation:

$$x_n = \frac{x_{-1}x_{-2}x_{-3}}{x_{-2}x_{-3}s_{n+1} + x_{-1}x_{-3}s_{n-1} + x_{-1}x_{-2}s_n}, \quad n \geq -3, \tag{10}$$

where $(s_n)_{n\geq-3}$ is the solution to the following difference equation:

$$s_n = s_{n-1} + s_{n-3}, \quad n \in \mathbb{N}_0, \tag{11}$$

satisfying the conditions in (9) (s_{-3} is naturally calculated from (8) with $n = 0$).

Here we also explain how a representation formula for general solution to an equation including representations (7) and (10) of equations (5) and (6), respectively, can be obtained. A comment on the choice of the sequences $(s_n)_{n\geq-3}$ defined by (8) and (9), i.e., by (11) and (9), is also given. Finally, we give a detailed explanation how the corresponding representation for a related nonlinear fourth-order difference equation can be obtained.

2 Main results

In this section we prove our main results. We consider three classes of difference equations separately. The first one extends equation (2), the second extends equations (5) and (6), while the third is their fourth-order relative.

2.1 A class of nonlinear second order difference equations

Let $f : \mathcal{D}_f \to \mathbb{R}$ be a "1–1" continuous function on its domain $\mathcal{D}_f \subseteq \mathbb{R}$. Here we consider the following second-order difference equation:

$$x_n = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2})\big), \quad n \geq 2, \tag{12}$$

where parameters a, b and initial values x_0 and x_1 are real numbers.

Since f is "1–1", from (12) we have

$$f(x_n) = af(x_{n-1}) + bf(x_{n-2}), \quad n \geq 2. \tag{13}$$

By using the change of variables

$$y_n = f(x_n), \quad n \in \mathbb{N}_0, \tag{14}$$

equation (13) is transformed to the following one:

$$y_n = ay_{n-1} + by_{n-2}, \quad n \geq 2. \tag{15}$$

If $b = 0$, then (15) becomes

$$y_n = ay_{n-1}, \quad n \geq 2,$$

from which it follows that

$$y_n = a^{n-1} y_1, \quad n \in \mathbb{N}. \tag{16}$$

Using the change of variables (14) in (16), as well as the assumption that f is "1–1", we obtain

$$x_n = f^{-1}\left(a^{n-1} f(x_1)\right), \quad n \in \mathbb{N}.$$

If $b \neq 0$, then equation (15) is homogeneous linear second-order difference equation with constant coefficients, so is solvable.

If $a^2 + 4b \neq 0$, then it is known that the solution to equation (15) with initial values y_0 and y_1 is given by

$$y_n = \frac{(y_0\lambda_2 - y_1)\lambda_1^n + (y_1 - y_0\lambda_1)\lambda_2^n}{\lambda_2 - \lambda_1}, \tag{17}$$

for $n \in \mathbb{N}_0$ (see [2]), where

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{a - \sqrt{a^2 + 4b}}{2}. \tag{18}$$

Formula (17) can be written in the following form:

$$y_n = by_0 \frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} + y_1 \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}_0. \tag{19}$$

Let $(s_n)_{n\in\mathbb{N}_0}$ be the solution to equation (15) such that

$$s_0 = 0 \quad \text{and} \quad s_1 = 1. \tag{20}$$

Then, it is easy to see that

$$s_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}_0. \tag{21}$$

By using (21) in equality (19), we obtain the following known representation of solutions to equation (15) (see [59], and also [14]):

$$y_n = by_0 s_{n-1} + y_1 s_n, \tag{22}$$

for $n \in \mathbb{N}_0$.

It should be noted here that formula (22) really holds for $n = 0$. Indeed, since $b \neq 0$, equation (15) can be prolonged for negative indices in a natural way. In particular, for the solution s_n we have

$$s_{n-2} = \frac{s_n - as_{n-1}}{b}. \tag{23}$$

If we put $n = 1$ in (23) and use initial conditions (20), we obtain

$$s_{-1} = \frac{1}{b}. \tag{24}$$

Note also that recurrence relation (23) enables us to define sequence s_n for every $n \in \mathbb{Z}$.

Using (14) in (22), as well as the assumption that f is "1–1", we obtain

$$x_n = f^{-1}\big(bf(x_0)s_{n-1} + f(x_1)s_n\big),$$

for $n \in \mathbb{N}_0$.

If $a^2 + 4b = 0$, then it is known that the solution to equation (15) with initial values y_0 and y_1 is given by

$$y_n = \big(y_1 n + \lambda_1 y_0(1-n)\big)\lambda_1^{n-1}, \tag{25}$$

for $n \in \mathbb{N}_0$, where

$$\lambda_1 = a/2.$$

On the other hand, we have that

$$s_n = n\lambda_1^{n-1}, \quad n \in \mathbb{N}_0. \tag{26}$$

Using (26) in (25), we obtain that the representation of solutions to equation (15) given in (22) holds also in this case.

Remark 1 When $b = 0$, equation (12) is of the first order. Thus to determine a concrete solution to the equation, it is necessary and sufficient only to know the value of x_1.

If we want to find the solution $(s_n)_{n\in\mathbb{N}_0}$ to equation

$$y_n = ay_{n-1}, \quad n \in \mathbb{N} \tag{27}$$

(note that here the domain $n \geq 2$ is prolonged) satisfying initial conditions (20), then we would have

$$1 = s_1 = as_0 = 0,$$

which is not possible.

Hence, it makes no sense to talk about solutions to equation (27) satisfying initial conditions (20).

However, instead of such a solution, it is natural to consider the solution $(s_n)_{n\in\mathbb{N}}$ to equation (27) satisfying the following condition:

$$s_1 = 1, \tag{28}$$

which is obviously equal to

$$s_n = a^{n-1}, \quad n \in \mathbb{N}.$$

From the above consideration we see that the following theorem holds.

Theorem 1 *Consider equation (12), where parameters a, b and initial values x_0 and x_1 are real numbers, and $f : \mathcal{D}_f \to \mathbb{R}$ is a "1–1" continuous function. Then the following statements hold:*

(a) *Assume that $b = 0$, and let $(s_n)_{n\in\mathbb{N}}$ be the solution to equation (15) satisfying initial condition (28). Then every well-defined solution to equation (12) has the following representation:*

$$x_n = f^{-1}\big(s_n f(x_1)\big), \tag{29}$$

for $n \in \mathbb{N}$.

(b) *Assume that $b \neq 0$, and let $(s_n)_{n\in\mathbb{N}_0}$ be the solution to equation (15) satisfying initial conditions (20). Then every well-defined solution to equation (12) has the following representation:*

$$x_n = f^{-1}\big(bf(x_0)s_{n-1} + f(x_1)s_n\big), \tag{30}$$

for $n \in \mathbb{N}_0$.

Now we give a few applications of Theorem 1.

Example 1 Let

$$f(t) = \frac{1}{t}. \tag{31}$$

Then, $\mathcal{D}_f = \mathbb{R} \setminus \{0\}$, whereas equation (12) becomes

$$x_n = \left(\frac{a}{x_{n-1}} + \frac{b}{x_{n-2}}\right)^{-1} = \frac{x_{n-2}x_{n-1}}{ax_{n-2} + bx_{n-1}}, \quad n \geq 2. \tag{32}$$

Here we can also assume that parameters a, b and initial values x_0 and x_1 are complex numbers, since function (31) is "1–1" on $\mathbb{C} \setminus \{0\}$.

First, note that $\mathcal{D}_f$ is not the whole real line, and that on the domain, function (31) is an involution, that is,

$$f^{-1}(t) = f(t), \quad t \in \mathcal{D}_f.$$

If $b = 0$, then by Theorem 1(a) we see that formula (29) holds. Using (31) in (29), we obtain that general solution to equation (32) in this case is

$$x_n = f^{-1}\big(s_n f(x_1)\big) = \left(\frac{s_n}{x_1}\right)^{-1} = \frac{x_1}{a^{n-1}}, \tag{33}$$

for $n \in \mathbb{N}$.

If $b \neq 0$, then by Theorem 1(b) we see that formula (30) holds. Using (31) in (30), we obtain that general solution to equation (32) in this case is

$$\begin{aligned} x_n &= f^{-1}\big(bf(x_0)s_{n-1} + f(x_1)s_n\big) \\ &= \left(\frac{b}{x_0}s_{n-1} + \frac{1}{x_1}s_n\right)^{-1} \\ &= \frac{x_0x_1}{bx_1s_{n-1} + x_0s_n}, \end{aligned} \tag{34}$$

for $n \in \mathbb{N}_0$.

If $a = b = 1$, then equation (15) is

$$y_n = y_{n-1} + y_{n-2}, \quad n \geq 2, \tag{35}$$

which means that the solution to equation (35) satisfying initial conditions (20) is the Fibonacci sequence. Hence, from formula (34) we obtain that general solution to difference equation (2) is given by formula (3).

This is a theoretical explanation how representation formula (3) for well-defined solutions to equation (2) can be obtained, in an elegant and constructive way. Formula (3) was presented in [58, Theorem 4.1], where it was proved by induction, and no theory behind it was given therein.

The following example is a natural generalization of Example 1 whose solution is not a rational function of the initial values and sequence s_n defined above.

Example 2 Let

$$\widetilde{f}_k(t) = t^{2k+1}, \quad k \in \mathbb{Z}. \tag{36}$$

Then, $\mathcal{D}_{\widetilde{f}_k} = \mathbb{R}$ if $k \in \mathbb{N}_0$, $\mathcal{D}_{\widetilde{f}_k} = \mathbb{R} \setminus \{0\}$ if $k \in \mathbb{Z} \setminus \mathbb{N}_0$, whereas equation (12) becomes

$$x_n = \big(ax_{n-1}^{2k+1} + bx_{n-2}^{2k+1}\big)^{\frac{1}{2k+1}}, \quad n \geq 2. \tag{37}$$

It is clear that $\widetilde{f}_k$ is "1–1" continuous function on $\mathcal{D}_{f_k}$ for each $k \in \mathbb{Z}$.

If $b = 0$, then by Theorem 1(a) we see that formula (29) holds. Using (36) in (29), we obtain that general solution to equation (37) in this case is

$$x_n = \widetilde{f}_k^{-1}\big(s_n\widetilde{f}_k(x_1)\big) = \big(s_nx_1^{2k+1}\big)^{\frac{1}{2k+1}} = a^{\frac{n-1}{2k+1}}x_1, \tag{38}$$

for $n \in \mathbb{N}$ (here we assume that $a \neq 0$, when $k \leq -1$).

If $b \neq 0$, then by Theorem 1(b) we see that formula (30) holds. Using (36) in (30), we obtain that general solution to equation (37) in this case is

$$\begin{aligned} x_n &= \widetilde{f}_k^{-1}\big(b\widetilde{f}_k(x_0)s_{n-1} + \widetilde{f}_k(x_1)s_n\big) \\ &= \big(bx_0^{2k+1}s_{n-1} + x_1^{2k+1}s_n\big)^{\frac{1}{2k+1}}, \end{aligned} \tag{39}$$

for $n \in \mathbb{N}_0$.

Note that if $k \in \mathbb{Z} \setminus \mathbb{N}_0$, then $-(2k+1) \geq 1$, and (39) can be written in the following form:

$$x_n = x_0 x_1 \left(b x_1^{-(2k+1)} s_{n-1} + x_0^{-(2k+1)} s_n\right)^{\frac{1}{2k+1}},$$

for $n \in \mathbb{N}_0$.

Theorem 1 says that when $b \neq 0$, every well-defined solution to (12) has representation (30) through the solution to equation (15) satisfying (20). Note that the choice of initial conditions, is, in fact, quite arbitrary. This means that instead of the solution one can try to find the corresponding representation through the solution to equation (15) with any initial values y_0 and y_1. It is clear that the representation is not always possible, since for $y_0 = y_1 = 0$ we obtain the trivial solution $y_n = 0$ to equation (15), for which $c_1 y_n + c_2 y_{n-1} = 0$ for every $n \in \mathbb{N}$, so that the solution cannot generate any other solution to the equation.

This observation and a result in [15] motivate us to find all possible initial values such that every solution to equation (15) can be represented in the following form:

$$y_n = c_1 s_{n+1}(\vec{v}) + c_2 s_n(\vec{v}), \quad n \in \mathbb{N}_0, \tag{40}$$

where $\vec{v} = (v_0, v_1)$ is a vector in $\mathbb{C}^2$, and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ is the solution to equation (15) such that

$$s_0 = v_0 \quad \text{and} \quad s_1 = v_1. \tag{41}$$

Note that representation (40) differs from (22) in that the indices of the sequences s_{n-1} and s_n are shifted forward by one, besides the choice of their initial values.

Theorem 2 *Assume that* $b \neq 0$ *and* $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ *is the solution to equation* (15) *such that* (41) *holds. Then representation* (40) *holds if and only if*

$$v_1^2 \neq v_0(a v_1 + b v_0). \tag{42}$$

Further, if (42) *holds, then*

$$y_n = \frac{(v_1 y_0 - v_0 y_1) s_{n+1}(\vec{v}) + (v_1 y_1 - a v_1 y_0 - b v_0 y_0) s_n(\vec{v})}{v_1^2 - a v_0 v_1 - b v_0^2}, \quad n \in \mathbb{N}_0, \tag{43}$$

for every solution $(y_n)_{n\in\mathbb{N}_0}$ *to equation* (15).

Proof Due to the linearity of equation (15), the sequence $(c_1 s_{n+1}(\vec{v}) + c_2 s_n(\vec{v}))_{n\in\mathbb{N}_0}$, is a solution to the equation for every $c_1, c_2 \in \mathbb{C}$.

It is clear that every solution to the equation can be written in the form (40) if and only if the following system:

$$c_1 s_1(\vec{v}) + c_2 s_0(\vec{v}) = y_0,$$
$$c_1 s_2(\vec{v}) + c_2 s_1(\vec{v}) = y_1$$

has a solution for all $y_0, y_1 \in \mathbb{C}$, that is, if the system

$$\begin{aligned} v_1c_1 + v_0c_2 &= y_0, \\ v_2c_1 + v_1c_2 &= y_1, \end{aligned} \tag{44}$$

has a solution for all $y_0, y_1 \in \mathbb{C}$, which is equivalent to

$$\Delta := \begin{vmatrix} v_1 & v_0 \\ v_2 & v_1 \end{vmatrix} = v_1^2 - v_2v_0 = v_1^2 - av_0v_1 - bv_0^2 \neq 0, \tag{45}$$

that is, to (42).

If (42) holds, then from (44) it follows that

$$c_1 = \frac{1}{\Delta}\begin{vmatrix} y_0 & v_0 \\ y_1 & v_1 \end{vmatrix} = \frac{v_1y_0 - v_0y_1}{v_1^2 - av_0v_1 - bv_0^2} \tag{46}$$

and

$$c_2 = \frac{1}{\Delta}\begin{vmatrix} v_1 & y_0 \\ v_2 & y_1 \end{vmatrix} = \frac{v_1y_1 - av_1y_0 - bv_0y_0}{v_1^2 - av_0v_1 - bv_0^2}. \tag{47}$$

From (40), (46) and (47), representation (43) easily follows, finishing the proof of the theorem. □

Remark 2 Condition (42) is not satisfied if and only if

$$v_1^2 = v_0(av_1 + bv_0). \tag{48}$$

If $v_0 = 0$ from (48) we have $v_1 = 0$, whereas if $v_1 = 0$ then $v_0 = 0$. If $v_0 \neq 0 \neq v_1$, then from (48) we have

$$\left(\frac{v_1}{v_0}\right)^2 = a\frac{v_1}{v_0} + b,$$

which is equivalent to

$$v_1 = \lambda_1 v_0 \quad \text{or} \quad v_1 = \lambda_2 v_0, \tag{49}$$

where λ_1 and λ_2 are defined in (18).

Hence, representation (40) is not possible for every solution to equation (15) if and only if (49) holds.

Corollary 1 *Assume that* $b \neq 0$ *and* $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ *is the solution to equation* (15) *such that* (41) *holds. Then the following representation holds*:

$$y_n = \widetilde{c}_1 s_{n+2}(\vec{v}) + \widetilde{c}_2 s_{n+1}(\vec{v}), \quad n \in \mathbb{N}_0, \tag{50}$$

if and only if (42) *holds.*

Further, if (42) *holds, then*

$$y_n = \frac{(v_1y_1 - av_1y_0 - bv_0y_0)s_{n+2}(\vec{v}) + (bv_1y_0 - bv_0y_1 - av_1y_1 + a^2v_1y_0 + abv_0y_0)s_{n+1}(\vec{v})}{b(v_1^2 - av_0v_1 - bv_0^2)}, \tag{51}$$

$n \in \mathbb{N}_0$, *for every solution* $(y_n)_{n\in\mathbb{N}_0}$ *to the equation.*

Proof By Theorem 2, representation (40) holds if and only if condition (42) holds. On the other hand, from (15) we have

$$s_n(\vec{v}) = \frac{s_{n+2}(\vec{v}) - as_{n+1}(\vec{v})}{b}, \tag{52}$$

for every $n \in \mathbb{N}_0$.

Using (52) in (40), we obtain

$$\begin{aligned} y_n &= c_1s_{n+1}(\vec{v}) + c_2\left(\frac{s_{n+2}(\vec{v}) - as_{n+1}(\vec{v})}{b}\right) \\ &= \left(c_1 - \frac{ac_2}{b}\right)s_{n+1}(\vec{v}) + \frac{c_2}{b}s_{n+2}(\vec{v}) \\ &= \widetilde{c}_1s_{n+2}(\vec{v}) + \widetilde{c}_2s_{n+1}(\vec{v}), \end{aligned} \tag{53}$$

for every $n \in \mathbb{N}_0$, which means that representation (50) holds.

If representation (50) holds, then by using

$$s_{n+2}(\vec{v}) = as_{n+1}(\vec{v}) + bs_n(\vec{v}) \tag{54}$$

in (50), we get

$$\begin{aligned} y_n &= \widetilde{c}_1\left(as_{n+1}(\vec{v}) + bs_n(\vec{v})\right) + \widetilde{c}_2s_{n+1}(\vec{v}) \\ &= (a\widetilde{c}_1 + \widetilde{c}_2)s_{n+1}(\vec{v}) + b\widetilde{c}_1s_n(\vec{v}) \\ &= c_1s_{n+1}(\vec{v}) + c_2s_n(\vec{v}), \end{aligned} \tag{55}$$

for every $n \in \mathbb{N}_0$, which means that representation (40) holds, from which along with the first part of Theorem 2 it follows that condition (42) holds.

Now assume that condition (42) holds. Then by Theorem 2 we have that for every solution $(y_n)_{n\in\mathbb{N}_0}$ to equation (15) representation (43) holds. Using relation (52) in (43) and after some simple calculations, representation (51) follows, completing the proof of the corollary. □

By using the same procedure as in Corollary 1, it can be proved that the following representation holds:

$$y_n = \widetilde{c}_1s_{n+3}(\vec{v}) + \widetilde{c}_2s_{n+2}(\vec{v}), \quad n \in \mathbb{N}_0, \tag{56}$$

if and only if condition (42) holds, and that when (42) holds, there are two constants α_3 and β_3 such that the following representation holds:

$$y_n = \frac{\alpha_3 s_{n+3}(\vec{v}) + \beta_3 s_{n+2}(\vec{v})}{b^2(v_1^2 - av_0v_1 - bv_0^2)}, \tag{57}$$

$n \in \mathbb{N}_0$, for every solution $(y_n)_{n\in\mathbb{N}_0}$ to equation (15).

Moreover, the following theorem holds.

Theorem 3 *Assume that $b \neq 0$, $k \in \mathbb{N}$ and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ is the solution to equation* (15) *such that* (41) *holds. Then*

$$y_n = \widetilde{c}_1 s_{n+k}(\vec{v}) + \widetilde{c}_2 s_{n+k-1}(\vec{v}), \quad n \in \mathbb{N}_0, \tag{58}$$

if and only if (42) *holds.*

Further, if (42) *holds, then there are two constants α_k and β_k such that*

$$y_n = \frac{\alpha_k s_{n+k}(\vec{v}) + \beta_k s_{n+k-1}(\vec{v})}{b^{k-1}(v_1^2 - av_0v_1 - bv_0^2)}, \quad n \in \mathbb{N}_0, \tag{59}$$

for every solution $(y_n)_{n\in\mathbb{N}_0}$ to the equation. The sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ satisfy the following recurrence relations:

$$\alpha_{k+1} = \beta_k, \qquad \beta_{k+1} = b\alpha_k - a\beta_k. \tag{60}$$

Proof We prove the theorem by induction. Case $k = 1$ was proved in Theorem 2, whereas case $k = 2$ was treated in Corollary 1, except for the relations in (60). From (43), we have

$$y_n = \frac{\alpha_1 s_{n+1}(\vec{v}) + \beta_1 s_n(\vec{v})}{v_1^2 - av_0v_1 - bv_0^2}, \quad n \in \mathbb{N}_0, \tag{61}$$

where

$$\alpha_1 := v_1y_0 - v_0y_1 \quad \text{and} \quad \beta_1 := v_1y_1 - av_1y_0 - bv_0y_0. \tag{62}$$

Using (52) in (61), we have

$$\begin{aligned} y_n &= \frac{\alpha_1 s_{n+1}(\vec{v}) + \beta_1 s_n(\vec{v})}{v_1^2 - av_0v_1 - bv_0^2} \\ &= \frac{\alpha_1 s_{n+1}(\vec{v}) + \beta_1(\frac{s_{n+2}(\vec{v}) - as_{n+1}(\vec{v})}{b})}{v_1^2 - av_0v_1 - bv_0^2} \\ &= \frac{\beta_1 s_{n+2}(\vec{v}) + (b\alpha_1 - a\beta_1)s_{n+1}(\vec{v})}{b(v_1^2 - av_0v_1 - bv_0^2)} \\ &= \frac{\alpha_2 s_{n+2}(\vec{v}) + \beta_2 s_{n+1}(\vec{v})}{b(v_1^2 - av_0v_1 - bv_0^2)}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{63}$$

where

$$\alpha_2 := \beta_1 \quad \text{and} \quad \beta_2 := b\alpha_1 - a\beta_1, \tag{64}$$

which is (60) with $k = 1$.

Now assume that the result was proved for a $k_0 \in \mathbb{N}$. Using (52) in the relation (58), which is obtained when k is replaced by k_0, we obtain

$$\begin{aligned} y_n &= \widetilde{c}_1 s_{n+k_0}(\vec{v}) + \widetilde{c}_2\left(\frac{s_{n+k_0+1}(\vec{v}) - a s_{n+k_0}(\vec{v})}{b}\right) \\ &= \frac{\widetilde{c}_2}{b} s_{n+k_0+1}(\vec{v}) + \left(\widetilde{c}_1 - \frac{a\widetilde{c}_2}{b}\right) s_{n+k_0}(\vec{v}) \\ &= \widehat{c}_1 s_{n+k_0+1}(\vec{v}) + \widehat{c}_2 s_{n+k_0}(\vec{v}), \end{aligned} \tag{65}$$

for every $n \in \mathbb{N}_0$, which means that representation (58) holds for $k = k_0 + 1$.

If representation (58) holds for $k = k_0 + 1$, then by using relation (54) in (58), we get

$$\begin{aligned} y_n &= \widetilde{c}_1\left(a s_{n+k_0}(\vec{v}) + b s_{n+k_0-1}(\vec{v})\right) + \widetilde{c}_2 s_{n+k_0}(\vec{v}) \\ &= (a\widetilde{c}_1 + \widetilde{c}_2) s_{n+k_0}(\vec{v}) + b\widetilde{c}_1 s_{n+k_0-1}(\vec{v}) \\ &= c_1 s_{n+k_0}(\vec{v}) + c_2 s_{n+k_0-1}(\vec{v}), \end{aligned} \tag{66}$$

for every $n \in \mathbb{N}_0$, which means that representation (58) holds for $k = k_0$, from which, along with the induction hypothesis, it follows that condition (42) holds.

Now assume that condition (42) holds. Then by the induction hypothesis we have that for every solution $(y_n)_{n\in\mathbb{N}_0}$ to equation (15) representation (59) holds for $k = k_0$. Using relation (52) in (59), we have

$$\begin{aligned} y_n &= \frac{\alpha_{k_0} s_{n+k_0}(\vec{v}) + \beta_{k_0} s_{n+k_0-1}(\vec{v})}{b^{k_0-1}(v_1^2 - a v_0 v_1 - b v_0^2)} \\ &= \frac{\alpha_{k_0} s_{n+k_0}(\vec{v}) + \beta_{k_0}\left(\frac{s_{n+k_0+1}(\vec{v}) - a s_{n+k_0}(\vec{v})}{b}\right)}{b^{k_0-1}(v_1^2 - a v_0 v_1 - b v_0^2)} \\ &= \frac{\beta_{k_0} s_{n+k_0+1}(\vec{v}) + (b\alpha_{k_0} - a\beta_{k_0}) s_{n+k_0}(\vec{v})}{b^{k_0}(v_1^2 - a v_0 v_1 - b v_0^2)} \\ &= \frac{\alpha_{k_0+1} s_{n+k_0+1}(\vec{v}) + \beta_{k_0+1} s_{n+k_0}(\vec{v})}{b^{k_0}(v_1^2 - a v_0 v_1 - b v_0^2)}, \end{aligned} \tag{67}$$

where

$$\alpha_{k_0+1} := \beta_{k_0} \quad \text{and} \quad \beta_{k_0+1} := b\alpha_{k_0} - a\beta_{k_0}. \tag{68}$$

From (63), (64), (67) and (68), and the method of induction, we see that (59) and (60) hold for every $k \in \mathbb{N}$, as claimed. □

Remark 3 From the recurrence relations in (60) we see that the sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ are two solutions of the following difference equation:

$$x_{k+1} + a x_k - b x_{k-1} = 0, \quad k \geq 2. \tag{69}$$

The roots of the characteristic polynomial associated to equation (69) are

$$t_1 = \frac{-a + \sqrt{a^2 + 4b}}{2} \quad \text{and} \quad t_2 = \frac{-a - \sqrt{a^2 + 4b}}{2}, \tag{70}$$

so that the general solution to the equation is

$$x_k = c_1 t_1^k + c_2 t_2^k, \quad k \in \mathbb{N}. \tag{71}$$

Using the initial conditions (62) and (64) in the following formulas:

$$\alpha_k = \frac{(\alpha_1 t_2 - \alpha_2)t_1^{k-1} + (\alpha_2 - \alpha_1 t_1)t_2^{k-1}}{t_2 - t_1}, \quad k \in \mathbb{N}, \tag{72}$$

and

$$\beta_k = \frac{(\beta_1 t_2 - \beta_2)t_1^{k-1} + (\beta_2 - \beta_1 t_1)t_2^{k-1}}{t_2 - t_1}, \quad k \in \mathbb{N}, \tag{73}$$

are obtained closed-form formulas for sequences $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$.

This means that for each fixed k the constants α_k and β_k in representation (59) can be determined explicitly.

2.2 A class of nonlinear third order difference equations

Let $f : \mathcal{D}_f \to \mathbb{R}$ be a "1–1" continuous function on its domain $\mathcal{D}_f \subseteq \mathbb{R}$. Here we consider the following third-order difference equation:

$$x_n = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2}) + cf(x_{n-3})\big), \quad n \in \mathbb{N}_0, \tag{74}$$

where parameters a, b, c and initial values x_{-3}, x_{-2} and x_{-1} are real numbers.

Since f is "1–1", from (74) we have

$$f(x_n) = af(x_{n-1}) + bf(x_{n-2}) + cf(x_{n-3}), \tag{75}$$

for $n \in \mathbb{N}_0$.

By using the change of variables (14), equation (75) is transformed to the following linear one:

$$y_n = ay_{n-1} + by_{n-2} + cy_{n-3}, \tag{76}$$

for $n \in \mathbb{N}_0$.

If $c = 0$, then equation (76) is reduced to equation (15). If $c \neq 0$, then the following representation formula for general solution to equation (76):

$$y_n = s_n y_0 + (s_{n+1} - as_n)y_{-1} + cs_{n-1}y_{-2}, \quad n \geq -2, \tag{77}$$

where the sequence $(s_n)_{n\geq -3}$ is the solution to equation (76) satisfying initial conditions (9), was proved in [19].

From (77) it easily follows that

$$y_n = s_{n+1}y_{-1} + (s_{n+2} - as_{n+1})y_{-2} + cs_n y_{-3}, \tag{78}$$

for $n \geq -3$.

Using (14) in (78), as well as the assumption that f is "1–1", we obtain

$$x_n = f^{-1}\big(s_{n+1}f(x_{-1}) + (s_{n+2} - as_{n+1})f(x_{-2}) + cs_n f(x_{-3})\big), \tag{79}$$

for $n \geq -3$, from which, together with the choice of the sequence s_n, it follows that

$$x_n = f^{-1}\big(s_{n+1}f(x_{-1}) + (bs_n + cs_{n-1})f(x_{-2}) + cs_n f(x_{-3})\big), \tag{80}$$

for $n \geq -3$.

Theorem 4 *Consider equation (74), where parameters a, b, c and initial values x_{-3}, x_{-2} and x_{-1} are real numbers, $c \neq 0$, and $f : \mathcal{D}_f \to \mathbb{R}$ is a "1–1" continuous function. Let $(s_n)_{n\geq -3}$ be the solution to equation (76) satisfying the conditions in (9), then every well-defined solution to equation (74) has representations (79) and (80).*

Example 3 Let f be given by (31). Then, equation (74) becomes

$$\begin{aligned} x_n &= \left(\frac{a}{x_{n-1}} + \frac{b}{x_{n-2}} + \frac{c}{x_{n-3}}\right)^{-1} \\ &= \frac{x_{n-1}x_{n-2}x_{n-3}}{ax_{n-2}x_{n-3} + bx_{n-1}x_{n-3} + cx_{n-1}x_{n-2}}, \end{aligned} \tag{81}$$

for $n \in \mathbb{N}_0$.

If $c \neq 0$, then by Theorem 4 we see that formula (79) holds, from which we obtain that general solution to equation (81) in this case is

$$\begin{aligned} x_n &= \left(\frac{s_{n+1}}{x_{-1}} + \frac{s_{n+2} - as_{n+1}}{x_{-2}} + c\frac{s_n}{x_{-3}}\right)^{-1} \\ &= \frac{x_{-3}x_{-2}x_{-1}}{x_{-2}x_{-3}s_{n+1} + x_{-1}x_{-3}(s_{n+2} - as_{n+1}) + cx_{-1}x_{-2}s_n}, \end{aligned} \tag{82}$$

for $n \geq -3$.

If $a = 0$ and $b = c = 1$, we obtain equation (5), so in this case from formula (82) we obtain formula (7). If $a = c = 1$ and $b = 0$, we get equation (6), so in this case, from formula (82) and the relation

$$s_{n+2} - as_{n+1} = cs_{n-1},$$

formula (10) is obtained.

These are theoretical explanations how representation formulas (7) and (10) from [58] for well-defined solutions to equations (5) and (6), respectively, can be obtained, in a constructive way. Since the function $f(t) = 1/t$ is "1–1" on $\mathbb{C} \setminus \{0\}$, note that formula (82) also holds if parameters a, b, c and initial values x_{-3}, x_{-2} and x_{-1} are complex numbers.

Now we will find all possible initial values such that every solution to equation (76) can be represented in the following form:

$$y_n = c_1 s_n(\vec{v}) + c_2 s_{n+1}(\vec{v}) + c_3 s_{n+2}(\vec{v}), \quad n \in \mathbb{N}_0, \tag{83}$$

where $\vec{v} = (v_0, v_1, v_2) \in \mathbb{C}^3$, and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ is the solution to the equation such that

$$s_0 = v_0, \qquad s_1 = v_1, \qquad s_2 = v_2. \tag{84}$$

Theorem 5 *Assume that $c \neq 0$ and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ is the solution to equation (76) such that (84) holds. Then representation (83) holds for every solution to the equation if and only if*

$$\begin{vmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & av_2 + bv_1 + cv_0 \\ v_2 & av_2 + bv_1 + cv_0 & (a^2 + b)v_2 + (ab + c)v_1 + acv_0 \end{vmatrix} \neq 0. \tag{85}$$

Proof Due to the linearity of equation (76), the sequence $(c_1 s_n(\vec{v}) + c_2 s_{n+1}(\vec{v}) + c_3 s_{n+2}(\vec{v}))_{n\in\mathbb{N}_0}$ is a solution to the equation for every $c_1, c_2, c_3 \in \mathbb{C}$.

It is clear that every solution to the equation can be written in the form (83) if and only if the following system:

$$\begin{aligned} c_1 s_0(\vec{v}) + c_2 s_1(\vec{v}) + c_3 s_2(\vec{v}) &= y_0, \\ c_1 s_1(\vec{v}) + c_2 s_2(\vec{v}) + c_3 s_3(\vec{v}) &= y_1, \\ c_1 s_2(\vec{v}) + c_2 s_3(\vec{v}) + c_3 s_4(\vec{v}) &= y_2 \end{aligned}$$

has a solution for all $y_0, y_1, y_2 \in \mathbb{C}$, that is, if the system

$$\begin{aligned} v_0 c_1 + v_1 c_2 + v_2 c_3 &= y_0, \\ v_1 c_1 + v_2 c_2 + v_3 c_3 &= y_1, \\ v_2 c_1 + v_3 c_2 + v_4 c_3 &= y_2 \end{aligned} \tag{86}$$

has a solution for all $y_0, y_1, y_2 \in \mathbb{C}$, which is equivalent to

$$\Delta := \begin{vmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \\ v_2 & v_3 & v_4 \end{vmatrix} \neq 0. \tag{87}$$

Using the facts

$$\begin{aligned} v_3 &= av_2 + bv_1 + cv_0, \\ v_4 &= \left(a^2 + b\right)v_2 + (ab + c)v_1 + acv_0 \end{aligned} \tag{88}$$

in (87), we have that it is equivalent to (85). □

Remark 4 If (85) holds, then from (86) it follows that

$$c_1 = \frac{1}{\Delta}\begin{vmatrix} y_0 & v_1 & v_2 \\ y_1 & v_2 & v_3 \\ y_2 & v_3 & v_4 \end{vmatrix}, \tag{89}$$

$$c_2 = \frac{1}{\Delta}\begin{vmatrix} v_0 & y_0 & v_2 \\ v_1 & y_1 & v_3 \\ v_2 & y_2 & v_4 \end{vmatrix}, \tag{90}$$

and

$$c_3 = \frac{1}{\Delta}\begin{vmatrix} v_0 & v_1 & y_0 \\ v_1 & v_2 & y_1 \\ v_2 & v_3 & y_2 \end{vmatrix}. \tag{91}$$

Using (88)–(91) in (83), we get a representation of solutions to equation (76) in terms of the initial values and the sequence $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$.

2.3 A class of nonlinear fourth order difference equations

Let $f : \mathcal{D}_f \to \mathbb{R}$ be a "1–1" continuous function on its domain $\mathcal{D}_f \subseteq \mathbb{R}$. Here we consider the following fourth-order difference equation

$$x_n = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2}) + cf(x_{n-3}) + df(x_{n-4})\big), \quad n \in \mathbb{N}_0, \tag{92}$$

where parameters a, b, c, d and initial values x_{-j}, $j = \overline{1,4}$, are real numbers.

Since f is "1–1", from (76) we have

$$f(x_n) = af(x_{n-1}) + bf(x_{n-2}) + cf(x_{n-3}) + df(x_{n-4}), \quad n \in \mathbb{N}_0. \tag{93}$$

By using the change of variables (14), equation (93) is transformed to the following linear one:

$$y_n = ay_{n-1} + by_{n-2} + cy_{n-3} + dy_{n-4}, \quad n \in \mathbb{N}_0. \tag{94}$$

If $d = 0$, then equation (92) is reduced to equation (76).

Assume that $d \neq 0$, and let $(s_n)_{n\in\mathbb{N}_0}$ be the solution to equation (94) satisfying the following conditions:

$$s_{-3} = s_{-2} = s_{-1} = 0, \qquad s_0 = 1. \tag{95}$$

Let

$$a_1 := a, \qquad b_1 := b, \qquad c_1 := c, \qquad d_1 := d. \tag{96}$$

Then, we have

$$y_n = a_1y_{n-1} + b_1y_{n-2} + c_1y_{n-3} + d_1y_{n-4}, \quad n \in \mathbb{N}_0. \tag{97}$$

Employing (94) where n is replaced by $n-1$ in (97), it follows that

$$\begin{aligned} y_n &= a_1y_{n-1} + b_1y_{n-2} + c_1y_{n-3} + d_1y_{n-4} \\ &= a_1(a_1y_{n-2} + b_1y_{n-3} + c_1y_{n-4} + d_1y_{n-5}) + b_1y_{n-2} + c_1y_{n-3} + d_1y_{n-4} \end{aligned}$$

$$
\begin{aligned}
&= (a_1a_1 + b_1)y_{n-2} + (b_1a_1 + c_1)y_{n-3} + (c_1a_1 + d_1)y_{n-4} + d_1a_1y_{n-5} \\
&= a_2y_{n-2} + b_2y_{n-3} + c_2y_{n-4} + d_2y_{n-5},
\end{aligned}
\tag{98}
$$

for $n \in \mathbb{N}$, where a_2, b_2, c_2, d_2 are defined by

$$
a_2 := a_1a_1 + b_1, \qquad b_2 := b_1a_1 + c_1, \qquad c_2 := c_1a_1 + d_1, \qquad d_2 := d_1a_1. \tag{99}
$$

Assume

$$
y_n = a_ky_{n-k} + b_ky_{n-k-1} + c_ky_{n-k-2} + d_ky_{n-k-3}, \tag{100}
$$

for a $k \ge 2$ and all n such that $n \ge k-1$, and

$$
\begin{aligned}
&a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1}, \\
&c_k = c_1a_{k-1} + d_{k-1}, \qquad d_k = d_1a_{k-1}.
\end{aligned}
\tag{101}
$$

Employing (97) where n is replaced by $n-k$, in (100), it follows that

$$
\begin{aligned}
y_n &= a_ky_{n-k} + b_ky_{n-k-1} + c_ky_{n-k-2} + d_ky_{n-k-3} \\
&= a_k(a_1y_{n-k-1} + b_1y_{n-k-2} + c_1y_{n-k-3} + d_1y_{n-k-4}) \\
&\quad + b_ky_{n-k-1} + c_ky_{n-k-2} + d_ky_{n-k-3} \\
&= (a_1a_k + b_k)y_{n-k-1} + (b_1a_k + c_k)y_{n-k-2} + (c_1a_k + d_k)y_{n-k-3} + d_1a_ky_{n-k-4} \\
&= a_{k+1}y_{n-k-1} + b_{k+1}y_{n-k-2} + c_{k+1}y_{n-k-3} + d_{k+1}y_{n-k-4},
\end{aligned}
\tag{102}
$$

for $n \ge k$, where $a_{k+1}, b_{k+1}, c_{k+1}, d_{k+1}$ are defined as follows:

$$
\begin{aligned}
&a_{k+1} := a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k, \\
&c_{k+1} := c_1a_k + d_k, \qquad d_{k+1} := d_1a_k.
\end{aligned}
\tag{103}
$$

From (98), (99), (102), (103) and the method of induction, we get that (100), (101) hold for every k and n such that $2 \le k \le n+1$.

Putting $k = n+1$ in (100) and using the relations in (101), it follows that

$$
\begin{aligned}
y_n &= a_{n+1}y_{-1} + b_{n+1}y_{-2} + c_{n+1}y_{-3} + d_{n+1}y_{-4} \\
&= a_{n+1}y_{-1} + (a_{n+2} - aa_{n+1})y_{-2} + (ca_n + da_{n-1})y_{-3} + da_ny_{-4},
\end{aligned}
\tag{104}
$$

for $n \in \mathbb{N}$.

From (101) we easily obtain that $(a_k)_{k\in\mathbb{N}}$ satisfies the difference equation (97). Besides, by choosing $k = 0$ in (103) we have

$$
a_1 = a_1a_0 + b_0, \qquad b_1 = b_1a_0 + c_0, \qquad c_1 = c_1a_0 + d_0, \qquad d_1 = d_1a_0.
$$

Since $d_1 = d \neq 0$, from these relations it follows that $a_0 = 1$, and $b_0 = c_0 = d_0 = 0$.

By choosing $k = -1, -2, -3$ in (103), it is obtained similarly that

$$a_{-1} = 0, \qquad b_{-1} = 1, \qquad c_{-1} = 0, \qquad d_{-1} = 0,$$
$$a_{-2} = 0, \qquad b_{-2} = 0, \qquad c_{-2} = 1, \qquad d_{-2} = 0,$$
$$a_{-3} = 0, \qquad b_{-3} = 0, \qquad c_{-3} = 0, \qquad d_{-1} = 1.$$

This means that $(a_n)_{n\ge-3}$ is the solution to (97) such that

$$a_{-3} = 0, \qquad a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1, \tag{105}$$

from which it follows that

$$a_n = s_n, \quad n \ge -3.$$

Hence, the following representation formula for general solution to equation (97) holds:

$$y_n = s_{n+1}y_{-1} + (s_{n+2} - as_{n+1})y_{-2} + (cs_n + ds_{n-1})y_{-3} + ds_n y_{n-4}, \tag{106}$$

for $n \ge -4$.

Using (14) in (106), as well as the assumption that f is "1–1", we obtain

$$x_n = f^{-1}\big(s_{n+1}f(x_{-1}) + (s_{n+2} - as_{n+1})f(x_{-2}) + (cs_n + ds_{n-1})f(x_{-3}) + ds_n f(x_{-4})\big), \tag{107}$$

for $n \ge -4$, from which together with the choice of the sequence s_n it follows that

$$\begin{aligned} x_n = f^{-1}\big(&s_{n+1}f(x_{-1}) + (bs_n + cs_{n-1} + ds_{n-2})f(x_{-2}) \\ &+ (cs_n + ds_{n-1})f(x_{-3}) + ds_n f(x_{-4})\big), \end{aligned} \tag{108}$$

for $n \ge -4$.

From the above considerations we see that the following theorem holds.

Theorem 6 *Consider equation* (92), *where parameters* a, b, c, d *and initial values* x_{-j}, $j = \overline{1,4}$, *are real numbers,* $d \ne 0$, *and* $f : \mathcal{D}_f \to \mathbb{R}$ *is a "1–1" continuous function. Let* $(s_n)_{n\ge-4}$ *be the solution to equation* (94) *satisfying the conditions in* (95), *then every well-defined solution to equation* (92) *has representations* (107) *and* (108).

Example 4 Let f be given by (31). Then, equation (92) becomes

$$\begin{aligned} x_n &= \left(\frac{a}{x_{n-1}} + \frac{b}{x_{n-2}} + \frac{c}{x_{n-3}} + \frac{d}{x_{n-4}}\right)^{-1} \\ &= \frac{x_{n-1}x_{n-2}x_{n-3}x_{n-4}}{ax_{n-2}x_{n-3}x_{n-4} + bx_{n-1}x_{n-3}x_{n-4} + cx_{n-1}x_{n-2}x_{n-4} + dx_{n-1}x_{n-2}x_{n-3}}, \end{aligned} \tag{109}$$

for $n \ge -4$.

If $d \neq 0$, then by Theorem 6 we see that formula (107) holds, from which we obtain that general solution to equation (109) in this case is

$$\begin{aligned} x_n &= \left(\frac{s_{n+1}}{x_{-1}} + \frac{s_{n+2} - a s_{n+1}}{x_{-2}} + \frac{c s_n + d s_{n-1}}{x_{-3}} + \frac{d s_n}{x_{-4}} \right)^{-1} \\ &= (x_{-1}x_{-2}x_{-3}x_{-4}) / \big(x_{-2}x_{-3}x_{-4}s_{n+1} + x_{-1}x_{-3}x_{-4}(s_{n+2} - a s_{n+1}) \\ &\quad + x_{-1}x_{-2}x_{-4}(c s_n + d s_{n-1}) + x_{-1}x_{-2}x_{-3} d s_{n-1} \big), \end{aligned} \tag{110}$$

for $n \geq -4$.

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