



## Pseudosymmetry Properties of Generalised Wintgen Ideal Legendrian Submanifolds

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**Abstract.** For Legendrian submanifolds  $M^n$  in Sasakian space forms  $\tilde{M}^{2n+1}(c)$ , I. Mihai obtained an inequality relating the normalised scalar curvature (intrinsic invariant) and the squared mean curvature and the normalised scalar normal curvature of  $M$  in the ambient space  $\tilde{M}$  (extrinsic invariants) which is called the *generalised Wintgen inequality*, characterising also the corresponding equality case. And a Legendrian submanifold  $M^n$  in Sasakian space forms  $\tilde{M}^{2n+1}(c)$  is said to be *generalised Wintgen ideal Legendrian submanifold* of  $\tilde{M}^{2n+1}(c)$  when it realises at everyone of its points the equality in such inequality. Characterisations based on some basic intrinsic symmetries involving the Riemann–Cristoffel curvature tensor, the Ricci tensor and the Weyl conformal curvature tensor belonging to the class of *pseudosymmetries in the sense of Deszcz* of such *generalised Wintgen ideal Legendrian submanifolds* are given.

### 1. Preliminaries

Curvature invariants are the main Riemannian invariants and the most natural ones. Among all curvature invariants, the most important are sectional, scalar and Ricci curvatures.

B. Y. Chen [2] established the inequality  $\rho \leq H^2 + c$ , for any submanifold  $M^n$  in a space form  $\tilde{M}^m(c)$ , whereby  $\rho = \frac{2\tau}{n(n-1)}$  is the normalised scalar curvature and  $H^2$  is the squared mean curvature of  $M^n$ . And equality in such inequality holds identically if and only if  $M^n$  is a totally umbilical submanifold.

For surfaces  $M^2$  in  $E^3$ , the *Euler inequality*  $K \leq H^2$ , whereby  $K$  is the Gauss curvature of  $M^2$  and  $H^2$  is the squared mean curvature of  $M^2$  in  $E^3$ . And,  $K = H^2$  everywhere on  $M^2$  if and only if the surface  $M^2$  is totally umbilical in  $E^3$ , i.e.  $k_1 = k_2$  at all points of  $M^2$ , or by a theorem of Meusnier, if and only if  $M^2$  is a part of a plane  $E^2$  or of a round sphere  $S^2$  in  $E^3$ . In 1979, P. Wintgen [25] proved that the Gauss curvature  $K$  and the squared mean curvature  $H^2$  and the normal curvature  $K^\perp$  of any surface  $M^2$  in  $E^4$  always satisfy the inequality  $K \leq H^2 - K^\perp$ , and that actually the equality holds if and only if the curvature ellipse of  $M^2$  in  $E^4$  is a circle. The Whitney 2–sphere satisfies identically the equality of the Wintgen inequality [20].

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2010 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C42

*Keywords.* Legendrian submanifold in Sasakian space forms, generalised Wintgen ideal Legendrian submanifold, pseudosymmetric space, Ricci pseudosymmetric space, manifold with pseudosymmetric Weyl conformal tensor, Choi–Lu plane of the generalised Wintgen ideal Legendrian submanifold

Received: 01 October 2018; Accepted: 25 February 2019

Communicated by Ljubica S. Velimirović

Research supported by the project 174012 of the Serbian Ministry of Education, Science and Technological Development

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A survey of recent results on surfaces satisfying identically the equality in Wintgen inequality is given in [5].

In 1999 De Smet, Dillen, Verstraelen and Vrancken [7] formulated the conjecture on Wintgen inequality (DDVV conjecture) for all submanifolds in all real space forms,

$$\rho \leq H^2 - \rho^\perp + c \tag{*}$$

whereby  $\rho$  is the normalized scalar curvature of  $M^n$  defined by

$$\rho = \frac{2}{n(n-1)} \sum_{i < j}^n \langle R(e_i, e_j)e_j, e_i \rangle,$$

and  $\rho^\perp$  is the normal scalar curvature function of  $M^n$  at a point  $p$ , defined by

$$\rho^\perp(p) = \frac{2}{n(n-1)} \sqrt{\sum_{i < j}^n \sum_{r < s}^2 \langle R^\perp(e_i, e_j) \xi_r, \xi_s \rangle^2},$$

$\{e_1, \dots, e_n\}$  is any orthonormal basis of the tangent space  $T_p(M^n)$  ( $p \in M^n$ ), and  $R$  is the Riemann–Christoffel curvature tensor of  $M^n$ , where  $R^\perp$  is the curvature tensor of the normal space, and  $\{\xi_1, \xi_2\}$  is an orthonormal basis of the normal space. They proved the above Wintgen inequality for all submanifolds  $M^n$  of codimension 2 in all real space forms  $\tilde{M}^{n+2}(c)$  of  $M^n$ , and also characterised the equality case in terms of the shape operators of  $M^n$  in  $\tilde{M}^{n+2}(c)$  [7].

Later, Choi and Lu [6], Lu [19] and Ge and Tang [14] proved that indeed (\*) holds in full generality for all submanifolds  $M^n$  in  $\tilde{M}^{n+m}(c)$  and gave a characterisation of the equality case in terms of an explicit description of the second fundamental form.

The submanifolds  $M^n$  in  $\tilde{M}^{n+m}(c)$  satisfying equality in Wintgen inequality are called *Wintgen ideal submanifolds*; for many examples and for geometrical properties of such submanifolds, see e.g. [3,6,19,20,21].

It should be observed that for submanifolds  $M^n$  in  $\tilde{M}^{n+m}(c)$  with flat normal connection, and thus in particular for hypersurfaces ( $m = 1$ ), the *Wintgen inequality* actually reduces to a *Chen inequality*  $\rho \leq \|H\|^2 + c$  and the corresponding ideal submanifolds  $M$  then are totally umbilical in  $\tilde{M}$  and hence spaces of constant curvature (and so are special Deszcz symmetric spaces, with  $L = 0$ ); and, as general basic reference for optimal inequalities relating various extrinsic and intrinsic characteristics of submanifolds, we refer to B.Y. Chen’s book ([4]).

We recall that an  $n$ -dimensional Riemannian manifold  $M^n$ , ( $n \geq 3$ ) is said to be a *pseudosymmetric space* in the sense of Deszcz or a *Deszcz symmetric space* ([8],[11],[16],[22],[23],[24]) if the (0, 6) tensors  $R \cdot R$  and  $Q(g, R)$  are linearly dependent at every point of  $M^n$ , i.e.  $R \cdot R = LQ(g, R)$  on  $\mathcal{U}_R = \{x \in M | R - (\frac{\kappa}{(n-1)n})G \neq 0 \text{ at } x\}$ , where  $L$  is some function on this set, called the *Deszcz sectional curvature function* of  $M^n$ . Hereby, by the action of the curvature operator  $R$  as a derivation on the (0, 4) curvature tensor  $R$  results the (0, 6) curvature tensor  $R \cdot R$ , i.e.,  $(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) = (R(X, Y) \cdot R)(X_1, X_2, X_3, X_4) = -R(R(X, Y)X_1, X_2, X_3, X_4) - R(X_1, R(X, Y)X_2, X_3, X_4) - R(X_1, X_2, R(X, Y)X_3, X_4) - R(X_1, X_2, X_3, R(X, Y)X_4)$ , and the Tachibana tensor  $Q(g, R)$  is the (0, 6) tensor which results from the action as a derivation on the (0, 4) curvature tensor  $R$ , by the metrical endomorphism, i.e.,  $Q(g, R) = \wedge_g \cdot R$ , or,  $Q(g, R)(X_1, X_2, X_3, X_4; X, Y) = (\wedge_g \cdot R)(X_1, X_2, X_3, X_4; X, Y) = ((X \wedge_g Y) \cdot R)(X_1, X_2, X_3, X_4) = -R((X \wedge_g Y)X_1, X_2, X_3, X_4) - R(X_1, (X \wedge_g Y)X_2, X_3, X_4) - R(X_1, X_2, (X \wedge_g Y)X_3, X_4) - R(X_1, X_2, X_3, (X \wedge_g Y)X_4)$ , whereby  $X, Y, X_1, X_2, X_3, X_4$  are arbitrary tangent vector fields on  $M$ .

The geometrical meaning of the tensor  $R \cdot R$  and the Tachibana tensor  $Q(g, R)$  is given in [15]. And a Riemannian manifold  $M^n$  of dimension  $n \geq 3$  is Deszcz symmetric or pseudosymmetric when its Deszcz sectional curvature or double sectional curvature function  $L(p, \pi, \bar{\pi})$  is *isotropic*, i.e., at all of its points  $p$  has the same value  $L(p)$  for all curvature dependent tangent planes  $\pi$  and  $\bar{\pi}$  at  $p$ . ([15])

The (0, 4) tensor  $R \cdot S$ , obtained by the action of the curvature operator  $R(X, Y)$  on the (0, 2) symmetric Ricci tensor  $S$ , given by  $(R \cdot S)(X_1, X_2; X, Y) = (R(X, Y) \cdot S)(X_1, X_2) = -S(R(X, Y)X_1, X_2) - S(X_1, R(X, Y)X_2)$ . The Ricci Tachibana tensor  $Q(g, S)$  is given by  $Q(g, S)(X_1, X_2; X, Y) = ((X \wedge_g Y) \cdot S)(X_1, X_2) = -S((X \wedge_g Y)X_1, X_2) - S(X_1, (X \wedge_g Y)X_2)$ .

The geometrical meaning of the tensor  $R \cdot S$  and the Tachibana  $Q(g, S)$  is given in [18].

A Riemannian manifold  $M^n$  ( $n \geq 3$ ), is said to be *Ricci pseudosymmetric space in the sense of Deszcz*, or *Ricci Deszcz symmetric space* if  $R \cdot S = L_S Q(g, S)$ , for some real valued function  $L_S$  on  $M$ . It is known that every Deszcz symmetric manifold automatically is Ricci Deszcz symmetric. The converse is not true in general ([8]).

A Riemannian manifold  $M^n$  ( $n \geq 4$ ), is said to have *pseudosymmetric Weyl conformal tensor* (see [9] and references therein) if the  $(0, 6)$  tensors  $C \cdot C$  and  $Q(g, C)$  satisfy the pseudosymmetry curvature condition  $C \cdot C = L_C Q(g, C)$ , for some function  $L_C : M \rightarrow R$  (on the open subset of  $M$  on which  $C \neq 0$ ). Hereby, by the action of the curvature operator  $C$  as a derivation on the  $(0, 4)$  curvature tensor  $C$  results the  $(0, 6)$  curvature tensor  $C \cdot C$ , i.e.,  $(C \cdot C)(X_1, X_2, X_3, X_4; X, Y) = (C(X, Y) \cdot C)(X_1, X_2, X_3, X_4) = -C(C(X, Y)X_1, X_2, X_3, X_4) - C(X_1, C(X, Y)X_2, X_3, X_4) - C(X_1, X_2, C(X, Y)X_3, X_4) - C(X_1, X_2, X_3, C(X, Y)X_4)$ , and the Weyl Tachibana tensor  $Q(g, C)$  is the  $(0, 6)$  tensor which results from the action as a derivation on the  $(0, 4)$  curvature tensor  $C$ , by the metrical endomorphism, i.e.,  $Q(g, C) = \wedge_g \cdot C$ , or,  $Q(g, C)(X_1, X_2, X_3, X_4; X, Y) = (\wedge_g \cdot C)(X_1, X_2, X_3, X_4; X, Y) = ((X \wedge_g Y) \cdot C)(X_1, X_2, X_3, X_4) = -C((X \wedge_g Y)X_1, X_2, X_3, X_4) - C(X_1, (X \wedge_g Y)X_2, X_3, X_4) - C(X_1, X_2, (X \wedge_g Y)X_3, X_4) - C(X_1, X_2, X_3, (X \wedge_g Y)X_4)$ , whereby  $X, Y, X_1, X_2, X_3, X_4$  are arbitrary tangent vector fields on  $M$ . The geometrical meaning of the tensor  $C \cdot C$  and the Weyl Tachibana tensor  $Q(g, C)$  is given in [17].

We mention that Chen ideal submanifolds in Euclidean spaces satisfying the presented above curvature conditions, as well as other conditions of this kind were studied among others in [10] and [13].

## 2. Generalised Wintgen inequality for Legendrian submanifolds

Let  $\tilde{M}^{2m+1}$  be a Riemannian manifold with  $\phi, \xi$  and  $\eta$  be tensor fields of type  $(1, 1), (1, 0)$  and  $(0, 1)$ , respectively. The triple  $(\phi, \xi, \eta)$  is called an *almost contact structure* if the following equalities are satisfied:

$$\eta(\xi) = 1, \eta(\phi(X)) = 0, \phi^2(X) = -X + \eta(X)\xi,$$

for any  $X \in T\tilde{M}^{2m+1}$ . The vector field  $\xi$  is called characteristic vector field.

If  $g$  is a pseudo-Riemannian metric on  $\tilde{M}^{2m+1}$ , then  $(\phi, \xi, \eta, g, \epsilon)$  is called an almost contact metric on  $\tilde{M}^{2m+1}$  if  $(\phi, \xi, \eta)$  is an almost contact structure such that

$$g(\xi, \xi) = \epsilon, \eta(X) = \epsilon g(\xi, X), X \in T\tilde{M}^{2m+1}, \epsilon = 1, \epsilon = -1$$

$$\eta(\phi(X), \phi(Y)) = g(X, Y) - \epsilon \eta(X)\eta(Y), X, Y \in T\tilde{M}^{2m+1}.$$

If this structure satisfies  $d\eta(X, Y) = g(\phi X, Y)$ , then this pseudo-Riemannian manifold with contact metric structure is called a *contact metric manifold*. A manifold  $M$  endowed with a normal contact metric structure  $(\phi, \xi, \eta, g, \epsilon)$  which satisfies  $(\tilde{\nabla}_X \phi)Y = \epsilon \eta(Y)X - g(X, Y)\xi, \tilde{\nabla}_X \xi = \phi X$ , for any vector fields  $X, Y$  on  $\tilde{M}^{2m+1}$ , whereby  $\tilde{\nabla}$  denotes the Riemannian connection with respect to  $g$ , is called a *Sasakian manifold*.

A plane section  $\pi$  in  $T\tilde{M}^{2m+1}$  is called a  $\phi$ -section if it is spanned by  $X$  and  $\phi(X)$ , whereby  $X$  is a unit tangent vector orthogonal to  $\xi$ . The sectional curvature of a  $\phi$ -section is called a  $\phi$ -sectional curvature. A Sasakian manifold with constant  $\phi$ -sectional curvature  $c$  is said to be a *Sasakian space form* and is denoted by  $\tilde{M}^{2m+1}(c)$ . The curvature tensor  $\tilde{R}$  of a Sasakian space form  $\tilde{M}^{2m+1}(c)$  is given by (see [1])

$$\begin{aligned} \tilde{R}(X, Y)Z &= \frac{\epsilon+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \\ &+ \frac{\epsilon-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - \\ &- g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi(X), Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

for any tangent vector fields  $X, Y, Z$  on  $\tilde{M}^{2m+1}(c)$ .

Let  $M^n$  be an  $n$ -dimensional submanifold in a Sasakian space form  $\tilde{M}^{2m+1}(c)$ . Denote by  $\nabla$  and  $h$  the Riemannian connection on  $M^n$  and the second fundamental form, respectively, then the Gauss equation is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

whereby  $R$  is the Riemann curvature tensor of  $M^n$ , and  $X, Y, Z, W$  are vectors tangent to  $M^n$ .

Let  $p \in M^n$  and  $\{e_1, e_2, \dots, e_n, \dots, e_{2m+1}\}$  is an orthonormal basis of the tangent space  $\tilde{M}^{2m+1}(c)$ , such that  $e_1, e_2, \dots, e_n$  are tangent to  $M^n$  at  $p$ . Then the mean curvature vector is given by

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i).$$

A submanifold  $M^n$  normal to  $\xi$  in a Sasakian manifold is said to be a  $C$ -totally real submanifold. In this case, it follows that  $\phi(T_p M^n) \subset T_p^\perp M^n$ , for every  $p \in M^n$ . In particular, if  $m = n$ , then  $M^n$  is called a Legendrian submanifold.

Let  $M^n$  be an  $n$ -dimensional Legendrian submanifold of a Sasakian space form  $\tilde{M}^{2n+1}(c)$  and  $\{e_1, e_2, \dots, e_n\}$  an orthonormal frame on  $M^n$  and  $\{e_{n+1}, \dots, e_{2n}, e_{2n+1} = \xi\}$  an orthonormal frame in the normal bundle  $T^\perp M^n$ .

Denote by  $h$  and  $A$  the second fundamental form and the shape operator of  $M^n$  in  $\tilde{M}^{2n+1}(c)$ . Then the Gauss equation is given by

$$R(X, Y, Z, W) = \frac{c+3}{4} \{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\} + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)).$$

In the recent paper [20], I. Mihai established a generalised Wintgen inequality for Legendrian submanifolds in Sasakian space forms.

**Theorem A.** ([20]). *Let  $M^n$  be an  $n$ -dimensional Legendrian submanifold of a Sasakian space form  $\tilde{M}^{2n+1}(c)$ . Then*

$$(\rho^\perp)^2 \leq \left( \|H\|^2 - \rho + \frac{c+3}{4} \right)^2 + \frac{4}{n(n-1)} \left( \rho - \frac{c+3}{4} \right) \frac{c-1}{4} + \frac{(c-1)^2}{8n(n-1)}, \tag{**}$$

and equality holds if and only if with respect to suitable orthonormal frames  $\{e_1, e_2, \dots, e_n\}$  and  $\{e_{n+1}, \dots, e_{2n}, e_{2n+1} = \xi\}$ , the shape operators of  $M^n$  in  $\tilde{M}^{2n+1}(c)$  are given by

$$A_{e_{n+1}} = \begin{pmatrix} \lambda_1 & \mu & 0 & \dots & 0 \\ \mu & \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_1 \end{pmatrix}, \tag{1}$$

$$A_{e_{n+2}} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 - \mu & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_2 \end{pmatrix}, \tag{2}$$

$$A_{e_{n+3}} = \begin{pmatrix} \lambda_3 & 0 & 0 & \dots & 0 \\ 0 & \lambda_3 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \lambda_3 \end{pmatrix}, \tag{3}$$

$$A_{e_{n+4}} = \dots = A_{e_{2n}} = A_{e_{2n+1}} = 0.$$

whereby  $\lambda_1, \lambda_2, \lambda_3$  and  $\mu$  are real functions on  $M^n$ .

Legendrian submanifolds  $M^n$  in Sasakian space forms  $\tilde{M}^{2n+1}(c)$  satisfying equality in generalised Wintgen inequality (\*\*) are called generalised Wintgen ideal Legendrian submanifolds. A frame  $\{e_1, e_2, \dots, e_n; e_{n+1}, \dots, e_{2n}, e_{2n+1}\}$  with the corresponding shape operators from Theorem A is called a Choi-Lu frame on such  $M^n$  in  $\tilde{M}^{2n+1}(c)$  and its distinguished tangent plane  $e_1 \wedge e_2$  is called the Choi-Lu plane of the generalised Wintgen ideal Legendrian submanifolds concerned.

### 3. Pseudosymmetry properties of generalised Wintgen ideal Legendrian submanifolds

First consider the *pseudosymmetry condition in the sense of Deszcz* of the generalised Wintgen ideal Legendrian submanifolds  $M^n$  in Sasakian space form  $\tilde{M}^{2m+1}(c)$ , ( $n \geq 4, m \geq 2$ ). Their Riemann-Christoffel curvature tensors are obtained by inserting the shape operators from Theorem A in the equation of Gauss. Up to the algebraic symmetries of the  $(0, 4)$  curvature tensor  $R$  of such generalised Wintgen ideal submanifolds, all components of  $R$  are zero except possibly the following ones

$$R_{1221} = 2\mu^2 - c_1, \tag{4}$$

$$R_{1kk1} = -\lambda_2\mu - c_1, \quad (k \geq 3) \tag{5}$$

$$R_{1kk2} = -\lambda_1\mu, \quad (k \geq 3) \tag{6}$$

$$R_{2kk2} = \lambda_2\mu - c_1, \quad (k \geq 3) \tag{7}$$

$$R_{kllk} = -c_1, \quad (k \neq l, k, l \geq 3), \tag{8}$$

where  $c_1 = \frac{c+3}{4} + \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ .

Then expressing this pseudosymmetry condition,  $R \cdot R = LQ(g, R)$ , to be satisfied by the  $(0, 6)$  tensors  $R \cdot R$  and  $Q(g, R)$  for some function  $L : M \rightarrow R$ , by evaluating these tensors on the tangent vectors  $\{e_1, e_2, \dots, e_n\}$ , one finds that this pseudosymmetry is characterised by the following system of algebraic equations

$$2\lambda_1\mu(2\mu^2 - c_1 - L) = 0 \tag{9}$$

$$2\lambda_2\mu(2\mu^2 - c_1 - L) = 0 \tag{10}$$

$$\lambda_1^2\mu^2 + \mu(2\mu + \lambda_2)(\lambda_2\mu - c_1 - L) = 0 \tag{11}$$

$$\lambda_1^2\mu^2 + \lambda_2\mu(\lambda_2\mu + c_1 + L) = 0 \tag{12}$$

$$\lambda_1\mu(c_1 + L) = 0 \tag{13}$$

$$\lambda_1^2\mu^2 + \lambda_2\mu(\lambda_2\mu - c_1 - L) = 0 \tag{14}$$

$$\lambda_1^2\mu^2 + \mu(\lambda_2 - 2\mu)(\lambda_2\mu + c_1 + L) = 0. \tag{15}$$

This system of equations is obtained by the evaluations of tensors  $R \cdot R$  and  $Q(g, R)$  on the following combinations of basic vectors  $\{e_1, \dots, e_n\}$ :  $(e_1, e_3, e_3, e_1; e_1, e_2), (e_1, e_3, e_3, e_2; e_1, e_2), (e_1, e_2, e_1, e_3; e_2, e_3), (e_1, e_4, e_3, e_4; e_1, e_3), (e_1, e_4, e_3, e_4; e_2, e_3), (e_2, e_4, e_3, e_4; e_2, e_3)$  and  $(e_1, e_2, e_2, e_3; e_1, e_3)$ , all other choices of combinations of vectors  $\{e_1, \dots, e_n\}$  leading either to one or other equation of the above system or to a triviality. And this system is satisfied if and only if (I)  $\mu = 0$ , in which case  $L = 0$ , or, (II)  $\mu \neq 0$  and  $\lambda_1 = \lambda_2 = 0$ , in which case  $L = -c_1 = -\frac{c+3}{4} - \lambda_3^2$ .

Therefore we obtained the following

**Theorem 1.** *A generalised Wintgen ideal Legendrian submanifold  $M^n$  of Sasakian space form  $\tilde{M}^{2n+1}(c)$ , ( $n \geq 4$ ), is a Deszcz symmetric Riemannian manifold if and only if it is totally umbilical (with  $L = 0$ ) or a minimal or pseudoumbilical submanifold ( $L = -\frac{c+3}{4} - H^2$ ) of this Sasakian space form  $\tilde{M}^{2n+1}(c)$ .*

Next, consider the *Ricci pseudosymmetry condition in the sense of Deszcz* of the generalised Wintgen ideal Legendrian submanifolds  $M^n$  in Sasakian space form  $\tilde{M}^{2n+1}(c)$ , ( $n \geq 4$ ). Using the nonzero components of Riemann-Christoffel curvature tensor  $R$ , the nontrivial components of the  $(0, 2)$  Ricci tensor of generalised Wintgen ideal Legendrian submanifolds  $M^n$  in Sasakian space form  $\tilde{M}^{2n+1}(c)$  in a tangent frame are found to be

$$S_{11} = 2\mu^2 - (n - 1)c_1 - (n - 2)\lambda_2\mu, \tag{16}$$

$$S_{22} = 2\mu^2 - (n - 1)c_1 + (n - 2)\lambda_2\mu, \tag{17}$$

$$S_{12} = -(n - 2)\lambda_1\mu, \tag{18}$$

$$S_{kk} = -(n - 1)c_1, (k \geq 3). \tag{19}$$

Then expressing the pseudosymmetry condition  $R \cdot S = L_S Q(g, S)$  satisfied by the  $(0, 4)$  tensors  $R \cdot S$  and  $Q(g, S)$  for some function  $L_S : M \rightarrow R$ , by evaluating these tensors on the tangent vectors  $\{e_1, e_2, \dots, e_n\}$ , one obtained that this Ricci pseudosymmetry in the sense of Deszcz is characterised by the following:

$$2(n-2)\lambda_1\mu(2\mu^2 - c_1 - L_S) = 0 \quad (20)$$

$$2(n-2)\lambda_2\mu(2\mu^2 - c_1 - L_S) = 0 \quad (21)$$

$$[-2\mu^2 + (n-2)\lambda_2\mu](-\lambda_2\mu - c_1 - L_S) - (n-2)\lambda_1^2\mu^2 = 0 \quad (22)$$

$$(n-2)\lambda_1\mu(\lambda_2\mu - c_1 - L_S) - \lambda_1\mu[-2\mu^2 + (n-2)\lambda_2\mu] = 0 \quad (23)$$

$$[-2\mu^2 - (n-2)\lambda_2\mu](\lambda_2\mu - c_1 - L_S) - (n-2)\lambda_1^2\mu^2 = 0. \quad (24)$$

And this system of equations is satisfied if and only if (I)  $\mu = 0, \lambda_1, \lambda_2, \lambda_3 \in R$  in which case  $L_S = 0$ , or (II)  $\mu \neq 0, \lambda_1 = \lambda_2 = 0, \lambda_3 \in R$  and  $L_S = -c_1 = -\frac{c+3}{4} - \lambda_3^2$ . Based on this result on Ricci Deszcz symmetry for Legendrian submanifolds  $M^n$  in Sasakian space form  $\tilde{M}^{2n+1}(c)$ , ( $n \geq 4$ ) and by virtue of Theorem 1, we thus obtained the following

**Theorem 2.** Any generalised Wintgen ideal Legendrian submanifold  $M^n$  in a Sasakian space form  $\tilde{M}^{2n+1}(c)$ , ( $n \geq 4$ ), is Deszcz symmetric if and only if it is Ricci Deszcz symmetric.

Similarly, in the following theorem, we characterised generalised Wintgen ideal Legendrian submanifolds in Sasakian space forms with pseudosymmetric Weyl conformal curvature tensor  $C$ , i.e satisfying the pseudosymmetry condition  $C \cdot C = L_C Q(g, C)$ , whereby  $Q(g, C)$  is the so-called Weyl–Tachibana curvature tensor. Hereby, the scalar curvature of generalised Wintgen ideal Legendrian submanifold  $M^n$  of Sasakian space form  $\tilde{M}^{2n+1}(c)$  is given by  $\tau = 4\mu^2 - n(n-1)c_1$  and  $\inf K = K_{12} = c_1 - 2\mu^2$ .

**Theorem 3.** Let  $M^n$ , ( $n \geq 4$ ) be a generalised Wintgen ideal Legendrian submanifold in a Sasakian space form  $\tilde{M}^{2n+1}(c)$ .

(i) Then  $M^n$  is conformally flat if and only if  $M^n$  is a totally umbilical submanifold in  $\tilde{M}^{2n+1}(c)$ .

(ii) If  $M^n$  is a non-conformally flat submanifold, then  $M^n$  has a pseudosymmetric Weyl conformal tensor  $C$  and corresponding function of pseudosymmetry is given by  $L_C = \frac{n-3}{(n-2)^2(n^2-1)}(\tau + n(n-1)\inf K)$ .

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