Polytopal Bier spheres and Kantorovich-Rubinstein polytopes of weighted cycles

Filip D. Jevtić
Mathematical Institute
SASA, Belgrade

Marinko Timotijević
Faculty of Science
University of Kragujevac

Rade T. Živaljević
Mathematical Institute
SASA, Belgrade

October 1, 2019

Abstract

The problem of deciding if a given triangulation of a sphere can be realized as the boundary sphere of a simplicial, convex polytope is known as the 'Simplicial Steinitz problem'. It is known by an indirect and non-constructive argument that a vast majority of Bier spheres are non-polytopal. Contrary to that, we demonstrate that the Bier spheres associated to threshold simplicial complexes are all polytopal. Moreover, we show that all Bier spheres are starshaped. We also establish a connection between Bier spheres and Kantorovich-Rubinstein polytopes by showing that the boundary sphere of the KR-polytope associated to a polygonal linkage (weighted cycle) is isomorphic to the Bier sphere of the associated simplicial complex of "short sets".

Keywords: Kantorovich-Rubinstein polytopes, Gale transform, Bier spheres, polyhedral combinatorics, simplicial Steinitz problem, polygonal linkages
MSC2010: 52B12, 52B35, 52B70

1 Introduction

The classic theory of the optimal transportation, as developed by L. Kantorovich [K42, Vi1, Vi2], is one of the pillars of the theory of linear programming [Ve2, Vi1]. The central paradigm of the theory is the Kantorovich duality principle [Vi1], in its manifold forms and incarnations. It includes, as one of the central consequences, the Kantorovich-Rubinstein theorem [Vi1, Theorem 1.14], which pertains to the case when the cost function is a metric.

Much more recent is the research program, proposed by A. Vershik in [Ve4], of studying "fundamental polytopes" or Kantorovich-Rubinstein polytopes as a tool for classifying metric spaces from the view point of polyhedral combinatorics (see Section 4 for an outline). These ideas can be traced back to Vershik's earlier publications [Ve2, Ve3], especially

This research was supported by the Grants 174020 and 174034 of the Ministry of Education, Science and Technological Development of the Republic of Serbia.

[MPV] (with J. Melleray and F. Petrov) and to Kantorovich himself, see [K-R] where the Kantorovich-Rubinstein norm $\|\mu - \nu\|_{KR}$ is introduced.

Bier spheres $Bier(K)$, where $K \subsetneq 2^{[n]}$ is an abstract simplicial complex, are combinatorially defined triangulations of the $(n-2)$-dimensional sphere S^{n-2} with interesting combinatorial and topological properties, [Lo1, M03]. These spheres are known to be shellable [BPSZ, ČD]. Moreover it is known, by an indirect and non-effective counting argument, that the majority of these spheres are non-polytopal, in the sense that they do not admit a convex polytope realization, see [M03, Section 5.6]. They also provide one of the most elegant proofs of the Van Kampen-Flores theorem [M03] and serve as one of the main examples of "Alexander complexes" [JNPZ].

Threshold complexes are ubiquitous in mathematics and arise, often in disguise and under different names, in areas as different as cooperative game theory (quota complexes and simple games) and algebraic topology of configuration spaces (polygonal linkages, complexes of short sets) [CoDe, Far, GaPa].

The following theorem establishes a connection between the boundary $\partial KR(d_L)$ of the Kantorovich-Rubinstein polytope of a weighted cycle, and the Bier sphere of the threshold complex of "short sets" of the associated polygonal linkage.

Theorem 1.1. *Suppose that $L = (l_1, l_2, \dots, l_n) \in \mathbb{R}^n_+$ is a strictly positive vector such that $\sum_{i=1}^n l_i = 1$. Let μ_L be the associated measure (weight distribution) on $[n]$ and let $\mathrm{Short}(L) = T_{\mu_L < 1/2} := \{I \subseteq [n] \mid \mu_L(I) < \frac{1}{2}\}$ be the associated simplicial complex of "short sets". We assume that L is generic in the sense that $(\forall I \subset [n])\ \mu_L(I) \neq \frac{1}{2}$. Let $\hat{L}$ be a weighted cycle (linkage) with bars of the length l_i and let d_L be the associated geodesic distance function on $[n]$. Then*

$$\partial KR(d_L) \cong Bier(\mathrm{Short}(L))\,. \tag{1.2}$$

The proof of Theorem 1.1, with all preliminary definitions and introductory facts, can be found in Section 4.

As a corollary of (1.2) we observe that the Bier spheres associated to complexes of generic short sets are always polytopal. In Section 2 (Theorem 2.3) we prove a more general result, somewhat surprising and interesting in itself, that the Bier spheres of any threshold complex (with an arbitrary quota and not necessarily generic) is polytopal. This should be compared to the fact (proven by an indirect and non-constructive argument) that a vast majority of Bier spheres are non-polytopal.

By an old result of Steinitz all triangulations of S^2 are polytopal and the problem of testing if a triangulation of a sphere is polytopal is known as the "Simplicial Steinitz problem". A closely related problem is the study of the asymptotic behavior of the number of nonisomorphic combinatorial types of triangulated (shellable, polytopal, starshaped, etc.) spheres with n-vertices. Early work of Goodman and Pollack [GP-1, GP-2], together with the estimates of Kalai [Ka], showed that asymptotically very few triangulated spheres are polytopal. Far reaching new results of this type, as well as a guide to some of the more recent publications, can be found in [B-Z, NSW, P-Z].

Recall that not all triangulations of $(n-2)$-dimensional spheres are starshaped in the sense that they admit a starshaped geometric realization in $\mathbb{R}^{n-1}$. An example of such a sphere can be found in [E, Theorem 5.5]. Our third main result, Theorem 3.5, claims that all Bier spheres (associated to all simplicial complexes $K \subsetneq 2^{[n]}$) are starshaped.

The notation and terminology in the paper is fairly standard. The book [E] is a general reference for the geometry of convex sets while the book [M03] provides an interesting and gentle introduction to combinatorial topology, with the emphasis on applications to combinatorics and discrete geometry.

2 Polytopal Bier spheres

The Alexander dual of a simplicial complex $K \subsetneq 2^{[n]}$ is the complex $K^\circ = \{I^c \mid I \notin K\}$. Suppose that $u_1 + u_2 + \cdots + u_n = 0$ is a 'minimal circuit' in $\mathbb{R}^{n-1}$, meaning that each proper subset of the collection of vectors $\{u_i\}_{i=1}^n$ is linearly independent. Suppose that $L = (l_1, l_2, \ldots, l_n) \in \mathbb{R}^n_+$ is a strictly positive vector. The associated measure (weight distribution) μ_L on $[n]$ is defined by $\mu_L(I) = \sum_{i\in I} l_i$ (for each $I \subseteq [n]$).

Given a threshold $\nu > 0$, the associated threshold complex is $T_{\mu_L<\nu} := \{I \subseteq [n] \mid \mu_L(I) < \nu\}$. Without loss of generality we assume that $\mu_L([n]) = l_1 + \cdots + l_n = 1$.

Remark 2.1. If $K = T_{\mu_L<\nu}$ is a threshold complex then $K = T_{\mu_L<\nu-\epsilon}$ for each sufficiently small $\epsilon > 0$. It follows that we may assume, without loss of generality, that $\mu_L(I) \neq \nu$ for each $I \subseteq [n]$ and, as a consequence, we may assume that the Alexander dual of K is $K^\circ = T_{\mu_L \leq 1-\nu} = T_{\mu_L<1-\nu}$.

For a simplicial complex $K \subset 2^{[n]}$ let K° its Alexander dual, and let $\Delta_S = \mathrm{Conv}\{e_i\}_{i\in S}$ be the geometric simplex spanned by $S \subseteq [n]$. Recall that for $K, L \subseteq 2^{[n]}$, the deleted join $K *_\Delta L$ is a subcomplex of the join $\Delta_{[n]} * \Delta_{[n]}$ defined by $K *_\Delta L := \{A \uplus B \mid A \in K, B \in L, A \cap B = \emptyset\}$.

For $K \subsetneq 2^{[n]}$, the associated Bier sphere is the deleted join,

$$Bier(K) := K *_\Delta K^\circ \subset \Delta_{[n]} *_\Delta \Delta_{[n]} \cong \partial\Diamond_{[n]} \tag{2.2}$$

where $\partial\Diamond_{[n]}$ is the boundary sphere of the n-dimensional cross-polytope $\Diamond_{[n]} = \mathrm{Conv}\{\pm e_i\}_{i=1}^n$.

Theorem 2.3. *$Bier(T_{\mu_L<\nu})$ is isomorphic to the boundary sphere of a convex polytope.*

Proof: Let $y_i = \frac{u_i}{l_i}$ which implies that $l_1y_1 + l_2y_2 + \cdots + l_ny_n = 0$ is (up to a multiplicative constant) the unique linear dependence of these vectors. Let $\alpha > 0$ a positive constant and $\beta = \frac{1}{\alpha}$.

Let $\Delta := \mathrm{Conv}\{y_i\}_{i=1}^n \subset \mathbb{R}^{n-1}$ be the simplex spanned by y_i and $\nabla_\alpha := -\alpha\Delta$ the simplex spanned by the vectors $-\alpha y_i$. We want to show that there exists $\alpha > 0$ such that the Bier sphere $Bier(T_{\mu_L<\nu})$ is isomorphic to the boundary sphere of the convex polytope,

$$Q_\alpha := \mathrm{Conv}(\Delta \cup \nabla_\alpha) = \mathrm{Conv}\{y_1, y_2, \ldots, y_n, -\alpha y_1, -\alpha y_2, \ldots, -\alpha y_n\}\,. \tag{2.4}$$

A linear transform of the collection of vectors (2.4), representing vertices of the polytope Q_α, is easily found and can be read off from the following matrix relation,

$$[y_1 \quad y_2 \quad \dots \quad y_n \quad -\alpha y_1 \quad -\alpha y_2 \quad \dots \quad -\alpha y_n] \begin{bmatrix} L^T & \alpha I_n \\ 0 & I_n \end{bmatrix} = 0 \tag{2.5}$$

where $L^T = (l_1, \dots, l_n)^T$ is a column vector and I_n the identity $(n \times n)$-matrix. If y is the row matrix $y = [y_1\, y_2\, \dots\, y_n]$ then the relation (2.5) can be rewritten as,

$$[y \quad -\alpha y] \begin{bmatrix} L^T & \alpha I_n \\ 0 & I_n \end{bmatrix} = 0\,. \tag{2.6}$$

Let $z : \mathbb{R}^{n-1} \to \mathbb{R}$ be a non-zero linear form such that the associated hyperplane $H_z := \{x \in \mathbb{R}^{n-1} \mid \langle z, x\rangle = 1\}$ is a supporting hyperplane of Q_α. The corresponding face of the polytope Q_α is described by a pair (I, J) of subsets of $[n]$ recording which vertices of the polytope Q_α belong to the hyperplane H_z. More explicitly

$$Q_\alpha \cap H_z = \mathrm{Conv}(\{y_i\}_{i\in I} \cup \{-\alpha y_j\}_{j\in J})\,. \tag{2.7}$$

It follows from (2.6) that,

$$[\langle z, y\rangle \quad \langle z, -\alpha y\rangle] \begin{bmatrix} L^T & \alpha I_n \\ 0 & I_n \end{bmatrix} = 0 \tag{2.8}$$

where $\langle z, y\rangle = [\langle z, y_1\rangle \dots \langle z, y_n\rangle]$. It follows from (2.7) that the ordered pair (I, J) of subsets of $[n]$ must satisfy the following:

$$(\forall i \in I)\, \langle z, y_i\rangle = 1 \qquad\qquad (\forall j \in J)\, \langle z, -\alpha y_j\rangle = 1 \tag{2.9}$$

$$(\forall k \notin I)\, \langle z, y_k\rangle < 1 \qquad\qquad (\forall k \notin J)\, \langle z, -\alpha y_k\rangle < 1 \tag{2.10}$$

From these relations it follows:

$$I \cap J = \emptyset \quad \text{and} \quad \emptyset \neq I \cup J \neq [n] \tag{2.11}$$

$$(\forall k \notin I \cup J)\; -\beta < \langle z, y_k\rangle < 1 \tag{2.12}$$

From (2.12) and the relation,

$$\sum_{i\in[n]} l_i \langle z, y_i\rangle = 0 = \sum_{i\in I} l_i - \sum_{j\in J} \beta l_j + \sum_{k\notin I\cup J} l_k \langle z, y_k\rangle \tag{2.13}$$

we deduce the following inequalities,

$$-\,\mu_L((I\cup J)^c) < \mu_L(I) - \beta\mu_L(J) < \beta\mu_L((I\cup J)^c)\,. \tag{2.14}$$

The inequalities (2.14) can be rewritten as follows,

$$\mu_L(J^c) > \beta\mu_L(J) \quad \text{and} \quad \mu_L(I) < \beta\mu_L(I^c)\,. \tag{2.15}$$

In light of the assumption $\sum_{i=1}^n l_i = 1$ we finally obtain the inequalities,

$$\mu_L(I) < \frac{\beta}{1+\beta} \quad \text{and} \quad \mu_L(J) < \frac{1}{1+\beta}\,. \tag{2.16}$$

In other words each face of the polytope Q_α, described by the equation (2.7), is associated a simplex $(I,J) \in Bier(T_{\mu_L<\nu})$ where $\nu = \frac{\beta}{1+\beta}$.

Conversely, let $(I,J) \in Bier(T_{\mu_L<\nu})$ be a face of the Bier sphere. Then the equation (2.16) is translated back to (2.14) and we can choose $\langle z, y_k\rangle$ (for $k \notin I \cup J$) satisfying (2.12) such that the equality (2.13) is also satisfied.

More explicitly, let $X = 1 - \mu_L(I) - \mu_L(J) = \sum_{i\notin I\cup J} l_i = \mu((I\cup J)^c)$. Then

$$-X < \mu_L(I) - \beta\mu_L(J) < \beta X \tag{2.17}$$

and there exists $\gamma \in (0,1)$ such that

$$\mu_L(I) - \beta\mu_L(J) = -\gamma X + (1-\gamma)\beta X \tag{2.18}$$

which is equivalent to

$$0 = \mu_L(I) - \beta\mu_L(J) + X(\gamma + (1-\gamma)(-\beta))\,. \tag{2.19}$$

Let us choose z such that $\langle z, y_k\rangle$ satisfies the equations (2.9) for $k \in I\cup J$ while for $k \notin I \cup J$

$$\langle z, y_k\rangle = \gamma \cdot 1 + (1-\gamma)(-\beta)\,. \tag{2.20}$$

This is possible in light of the equality (2.13). In turn this proves the validity of relations (2.9) and (2.10) and eventually leads to (2.7). This observation completes the proof of the theorem. $\square$

3 Starshaped Bier spheres

A d-dimensional triangulated sphere Σ^d is *starshaped* if there exists an embedding $e : \Sigma^d \to \mathbb{R}^{d+1}$, linear (affine) on simplices of Σ^d, and a point $c \in \mathbb{R}^{d+1} \setminus e(\Sigma^d)$, such that $[c,x] \cap [c,y] = \{c\}$ for each pair $x \neq y$ of distinct points in $e(\Sigma^d)$.

As shown by Ewald and Schulz in [E-S], for each $d \geq 4$ there exists a $(d-1)$-dimensional simplicial sphere which cannot be embedded in $\mathbb{R}^d$ as a starshaped set. An example of such a sphere is also described in [E, Theorem 5.5].

Surprisingly enough all Bier spheres turn out to be starshaped. As a consequence Bier spheres provide (at least statistically) numerous examples of non-polytopal, starshaped spheres. Indeed, according to [M03] there are more than $2^{(2^n/n)-2n^2}$ nonisomorphic Bier spheres, while the number of different combinatorial types of $(n-1)$-dimensional, simplicial convex polytopes with $2n$ vertices is not larger than 2^{4n^3}.

Note that an exponential upper bound to the number of starshaped sets in terms of the number of facets was recently proven in [AdBe, Theorem 2.5].

From here on we make a clear distinction between combinatorial, geometric, and topological (deleted) join of simplicial complexes, as emphasized and discussed in [M03, Section 4.2]. For example the "combinatorial deleted join" representation $\Delta_{[n]} *_\Delta \Delta_{[n]} \cong \partial\Diamond_{[n]}$, used in (2.2), naturally leads to a "geometric deleted join" representation

$$\Delta_e *_\Delta \Delta_{-e} = \partial\Diamond_{[n]} \tag{3.1}$$

where $\Delta_e = \mathrm{Conv}\{e_i\}_{i=1}^n$ and $\Delta_{-e} = \mathrm{Conv}\{-e_i\}_{i=1}^n$. More generally, for each (labeled) set $b = \{b_i\}_{i=1}^n$ of affinely independent vectors, there is an associated geometric simplex $\Delta_b = \mathrm{Conv}\{b_i\}_{i=1}^n$. Moreover, if $S \in K \subseteq 2^{[n]}$ is a simplex in an abstract simplicial complex, then the associated b-realization is the geometric simplex $R_b(S) = \mathrm{Conv}\{b_i\}_{i\in S}$.

For example if $\delta = (\delta_1, \ldots, \delta_n)$ is defined by $\delta_i = e_i - \frac{u}{n}$, where $u = e_1 + \cdots + e_n$, then

$$\Delta_\delta = \mathrm{Conv}\{\delta_i\}_{i=1}^n \quad \text{and} \quad \Delta_{-\delta} = \mathrm{Conv}\{-\delta_i\}_{i=1}^n\,. \tag{3.2}$$

If $T \subseteq [n]$ then $\overline{T}$ is the corresponding subset of $[\bar{n}] = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. The 'tautological geometric realization' of the abstract simplicial complex $Bier(K) = K *_\Delta K^\circ \subset 2^{[n]} * 2^{[\bar{n}]}$ is the geometric simplicial complex

$$\mathcal{R}_{\pm e}(Bier(K)) = \{R_e(S) * R_{-e}(T) \mid (S,T) \in K *_\Delta K^\circ\} \tag{3.3}$$

where $R_e(S) * R_{-e}(T) = \mathrm{Conv}(R_e(S) \cup R_{-e}(T)) \subset \partial\Diamond_{[n]}$ is the geometric join of simplices. Similarly, we define the 'canonical geometric realization' $\mathcal{R}_{\pm\delta}(Bier(K))$ by replacing e and $-e$ in (3.3) respectively by δ and $-\delta$. By construction

$$\mathcal{R}_{\pm\delta}(Bier(K)) = \pi(\mathcal{R}_{\pm e}(Bier(K))) \subset H_0 \tag{3.4}$$

where $\pi : \mathbb{R}^n \to H_0 := \{x \in \mathbb{R}^n \mid \langle u, x\rangle = 0\}$ is the orthogonal projection.

It remains to be shown that $\mathcal{R}_{\pm\delta}(Bier(K))$ is indeed a geometric realization of the abstract simplicial complex $Bier(K)$ and that it is precisely the desired starshaped realization.

Theorem 3.5. *Let $K \subsetneq 2^{[n]}$ be a simplicial complex and let $Bier(K)$ be the associated Bier sphere. Then $\mathcal{R}_{\pm\delta}(Bier(K))$ is a geometric realization of the abstract simplicial complex $Bier(K)$ which is starshaped as a subset of $H_0 := \{x \in \mathbb{R}^n \mid \langle u, x\rangle = 0\}$.*

Proof: Let $\mathrm{cone}(C) = \cup_{\lambda \geq 0}\, \lambda C$ be the convex cone with the apex at the origin generated by a convex set $C \subset H_0$. The theorem will follow from the observation that the collection of convex cones

$$\mathrm{Cone}_{\pm\delta}(K) = \{\mathrm{Cone}(R_\delta(S) * R_{-\delta}(T)) \mid (S,T) \in K *_\Delta K^\circ\} \tag{3.6}$$

is a complete simplicial fan in H_0. Recall [E, Chapter III] that a family Σ of simplicial cones (in V) with apex 0 is a complete simplicial fan if Σ is a covering of V and for each two cones $C_1, C_2 \in \Sigma$ the intersection $C_1 \cap C_2$ is their common face which is also an element in Σ.

We establish that (3.6) is a complete fan by showing that the associated geometric "shore subdivision" $\mathrm{Shore}_{\pm\delta}(K)$ (see [Lo2, Section 4.3]), obtained by the shore subdivision of each $\mathrm{cone}(R_\delta(S) * R_{-\delta}(T)) \in \mathrm{Cone}_{\pm\delta}(K)$, coincides with the fan Σ generated by the barycentric subdivision of the boundary of the simplex Δ_δ.

In the sequel we denote by $\nu(\Delta)$ the barycenter of a geometric simplex Δ. If $\Delta = R_b(S)$ we also write $\nu_b(S) := \nu(R_b(S))$. By definition each cone $C \in \mathrm{Shore}_{\pm\delta}(K)$, subdividing $\mathrm{cone}(R_\delta(S) * R_{-\delta}(T))$, is positively spanned by the vectors

$$\nu_\delta(S_1), \ldots, \nu_\delta(S_p), \nu_{-\delta}(T_q), \nu_{-\delta}(T_{q-1}), \ldots, \nu_{-\delta}(T_1) \tag{3.7}$$

where

$$S_1 \subset S_2 \subset \cdots \subset S_p \subseteq S \quad \text{ and } \quad T_1 \subset T_2 \subset \cdots \subset T_q \subseteq T\,. \tag{3.8}$$

On the other hand the cone spanned by (3.7) coincides with the cone positively spanned by the vectors

$$\nu_\delta(S_1), \ldots, \nu_\delta(S_p), \nu_\delta(T_q^c), \nu_\delta(T_{q-1}^c), \ldots, \nu_\delta(T_1^c) \tag{3.9}$$

(where $T_j^c = 2^{[n]} \setminus T_j$). In light of the fact that $(S, T) \in K *_\Delta K^\circ$, the condition (3.8) is equivalent to the condition

$$S_1 \subset S_2 \subset \cdots \subset S_p \subseteq S \subset T^c \subseteq T_q^c \subset T_{q-1}^c \subset \cdots \subset T_1 \tag{3.10}$$

This is precisely the condition that the positive span of (3.10) is a cone in Σ. $\square$

The reader familiar with [Lo1] (see also [Lo2]) will agree that the proof of Theorem 3.5 can be concisely described as a geometrization of the short and elegant proof of Mark de Longueville that $Bier(K)$ triangulates a sphere. Note however that the very existence of a canonical starshaped realization $\mathcal{R}_{\pm\delta}(Bier(K))$ of $Bier(K)$ is interesting in itself and have some interesting consequences. For example it allows to compare Bier spheres by the volume of the associated starshaped body

$$\mathrm{Star}(K) = \{\lambda x \in H_0 \mid x \in \mathcal{R}_{\pm\delta}(Bier(K)) \text{ and } 0 \le \lambda \le 1\}\,. \tag{3.11}$$

Moreover, it allows us to give a geometric interpretation of the classification of autodual simplicial complexes described in [Tim].

4 Kantorovich-Rubinstein polytopes

Let (X, ρ), $|X| = n$, be a finite metric space and let $V(X) := \mathbb{R}^X \cong \mathbb{R}^n$ be the associated vector space of real valued functions (weight distributions, signed measures) on X. Let $V_0(X) := \{\mu \in V(X) \mid \mu(X) = 0\}$ be the vector subspace of measures with total mass equal to zero, and $\Delta_X := \{\mu \in V(X) \mid \mu(X) = 1 \text{ and } (\forall x \in X)\, \mu(\{x\}) \geqslant 0\}$ the simplex of probability measures.

Let $\mathcal{T}_\rho(\mu, \nu)$ be the cost of the optimal transportation of measure μ to measure ν, where the cost of transporting the unit mass from x to y is $\rho(x, y)$. Then, as shown in [Ve2, Vi1], there exists a norm $\|\cdot\|_{KR}$ on $V_0(X)$ (called the Kantorovich-Rubinstein norm), such that,

$$\mathcal{T}_\rho(\mu, \nu) = \|\mu - \nu\|_{KR},$$

for each pair of probability measures $\mu, \nu \in \Delta_X$.

Definition 4.1. *The Kantorovich-Rubinstein polytope $KR(\rho)$, associated to a finite metric space (X, ρ), is the unit ball of the KR-norm in $V_0(X)$,*

$$KR(\rho) = \{x \in V_0(X) \mid \|x\|_{KR} \leqslant 1\}. \tag{4.2}$$

The following explicit description for $KR(\rho)$ can be deduced from the Kantorovich-Rubinstein theorem (Theorem 1.14 in [Vi1]),

$$KR(\rho) = \mathrm{Conv}\left\{\frac{e_x - e_y}{\rho(x,y)} \mid x, y \in X\right\}, \tag{4.3}$$

where $\{e_x\}_{x\in X}$ is the canonical basis in $\mathbb{R}^X$. More information about KR-polytopes can be found in [DH, GoPe, JJZ].

4.1 Metrics induced by weighted graphs

Let Γ be a simple graph on the set of vertices $V(\Gamma) = [n]$ with the set of edges $E(\Gamma) \subset 2^{[n]}$. We say that the graph Γ is positively weighted if we have chosen a positive weight function $w : E(\Gamma) \to \mathbb{R}_+$.

Definition 4.4. *Let $\Gamma = \Gamma([n], E(\Gamma), w)$ be a connected graph with a positive weight function $w : E(\Gamma) \to \mathbb{R}_+$. The associated "geodesic metric" d_Γ on $[n]$ is defined by*

$$d_\Gamma(i,j) = d_{i,j} = \min_{S\in\mathcal{P}_{ij}} \sum_{e\in S} w(e)\,, \tag{4.5}$$

where $\mathcal{P}_{ij}$ is the collection of all paths connecting vertices i and j.

Definition 4.4 is meaningless if the graph is not connected so in all subsequent statements we tacitly assume that Γ is a connected graph.

Lemma 4.6. *Let $([n], d_\Gamma)$ be the geodesic metric space induced by a positively weighted graph $\Gamma = ([n], E(\Gamma), w)$. Then*

$$KR(d_\Gamma) = \mathrm{Conv}\left(\{\pm v_{i,j}\}_{\{i,j\}\in E(\Gamma)}\right), \tag{4.7}$$

where $v_{i,j} = \frac{e_i - e_j}{d_{i,j}}$.

Proof: Assume that $\{k,l\} \notin E(\Gamma)$. Suppose that $S \in \mathcal{P}_{kl}$ is a path connecting k and l where the minimum in (4.5) is attained. By re-enumerating the vertices we may assume that $S = (\{k, k+1\}, \{k+1, k+2\}, \ldots, \{l-1, l\})$. Let

$$\alpha_i = \frac{d_{k+i,k+i+1}}{\sum_{j=0}^{l-k-1} d_{k+j,k+j+1}}, \text{ for } i \in \{0, \ldots, l-k-1\}.$$

Then $\sum_{j=0}^{l-k-1}\alpha_j=1$ and $v_{k,l}=\alpha_0 v_{k,k+1}+\ldots+\alpha_{l-k-1}v_{l-1,l}$ which completes the proof of the lemma. □

Let us recall that if $K=\mathrm{Conv}(X\cup\{\pm v\})$ is such that $0\in\mathrm{Conv}(X)$ and v is not in the vector subspace spanned by X, than K is a suspension over $\mathrm{Conv}(X)$,

$$K=\mathrm{Susp}\left(\mathrm{Conv}(X)\right).$$

As a direct corollary we get the following lemma.

Lemma 4.8. *Let Γ be a positively weighted graph, X subset of its vertices, and $x\in X$ such that $\Gamma|_{X^c\cup\{x\}}$ is connected, $\Gamma|_X$ is a non-trivial tree, and $E(\Gamma)=E(\Gamma|_{X^c\cup\{x\}})\cup E(\Gamma|_X)$. Then $KR(\Gamma)$ can be expressed as an iterated suspension,*

$$KR(\Gamma)=\mathrm{Susp}^{|X|-1}(KR(\Gamma|_{X^c\cup\{x\}})).$$

Proof. If $|X|=2$ we may assume without loss of generality that $X=\{n-1,n\}$ and that n is the isolated vertex. Since $v_{n-1,n}$ is not in the linear span of $KR(\Gamma|_{X^c\cup\{x\}})$, there is an isomorphism $KR(\Gamma)\cong\mathrm{Susp}(KR(\Gamma|_{X^c\cup\{x\}}))$. The general case of the lemma follows by induction on $|X|$. □

The following proposition is an immediate consequence of Lemma 4.8.

Proposition 4.9. *Let $T=([n],E(T),w)$ be a positively weighted tree on $[n]$. Then $KR(d_T)\cong\lozenge_{[n-1]}$. In other words, all KR-polytopes associated to weighted trees on $[n]$ are combinatorially equivalent.*

4.2 Proof of Theorem 1.1

Theorem 1.1 says, in a nutshell, that the boundary of the Kantorovich-Rubinstein polytope of the geodesic metric d_L coincides with the complex of short sets of the associated linkage $\hat{L}$. Here we obtain this result as a corollary of Theorem 2.3.

Proof of Theorem 1.1: Let Γ be a positively weighted cycle. More explicitly $\Gamma=(V(\Gamma),E(\Gamma),w)$ is a positively weighted graph where

$$V(\Gamma)=[n],E(\Gamma)=\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$$

and $w(\{i,i+1\})=l_i$ for each $i\in[n]$. (Here and to the end of this section we use the convention that $n+1:=1$.)

By Lemma 4.6,

$$KR(d_L)=\mathrm{Conv}\left\{\pm\frac{u_i}{l_i}\right\}_{i=1}^{n}=\mathrm{Conv}\{\pm y_i\}_{i=1}^{n} \tag{4.10}$$

where $y_i=\frac{u_i}{l_i}=\frac{e_{i+1}-e_i}{l_i}$. By comparison with (2.4) we observe that $\alpha=\beta=1$ and

$$KR(d_\Gamma)=Q_1=\mathrm{Conv}(\Delta\cup\nabla). \tag{4.11}$$

Finally by (2.16) we observe that

$$Q_1 = \text{Short}(L) := \{I \subseteq [n] \mid \mu_L(I) < 1/2\} \tag{4.12}$$

and the result follows as a consequence of Theorem 2.3. $\square$

It is known, see [Far], that moduli spaces M_L and M_{L^σ} of two linkages $L = (l_1, \ldots, l_n)$ and $L^\sigma = (l_{\sigma(1)}, \ldots, l_{\sigma(n)})$, where $\sigma \in \Sigma_n$ is a permutation, are homeomorphic. Since $\text{Short}(L) \cong \text{Short}(L^\sigma)$, it follows from Theorem 1.1 that the polytopes $KR(d_L)$ and $KR(d_{L^\sigma})$ are combinatorially isomorphic. Here we give a direct proof by constructing an explicit affine isomorphism.

Proposition 4.13. *Let* $L = (l_1, \ldots, l_n)$ *and* $L^\sigma = (l_{\sigma(1)}, \ldots, l_{\sigma(n)})$ *where* $\sigma \in \Sigma_n$ *is a permutation. Then* $KR(L) \cong KR(L^\sigma)$.

Proof. It is sufficient to prove the statement when σ is a cycle or transposition. If σ is a cycle, the statement is equivalent to relabeling the basis vectors. Let σ be a transposition. Without loss of generality, let $\sigma = (1, 2)$. Notice that there exists a hyperplane containing $e_4 - e_3, \ldots, e_n - e_{n-1}, e_1 - e_n$ which bisects the angle between $e_2 - e_1$ and $e_3 - e_2$. The reflection with respect to that hyperplane sends $KR(L)$ to $KR(L^\sigma)$. $\square$

Acknowledgements

It is our pleasure to acknowledge valuable remarks and useful suggestions by B. Sturmfels, G.M. Ziegler, S. Vrećica, D. Jojić, V. Grujić, members of Belgrade CGTA-seminar, and the anonymous referees. The authors were supported by the Grants 174020 and 174034 of the Ministry of Education, Science and Technological Development of Serbia.

References

[AdBe] K. Adiprasito and B. Benedetti. A Cheeger-type exponential bound for the number of triangulated manifolds, arXiv:1710.00130v2 [math.CO].

[B-Z] B. Benedetti and G.M. Ziegler. On locally constructible spheres and balls, *Acta Math.*, 206 (2011), 205–243.

[BPSZ] A. Björner, A. Paffenholz, J. Sjöstrand, G.M. Ziegler. Bier spheres and posets, *Discrete Comput. Geom.* 34 (2005), no. 1, 71–86.

[CoDe] R. Connelly and E.D. Demaine. *Geometry and topology of polygonal linkages*, Chapter 9 in *Handbook of Discrete and Computational Geometry* (third edition), CRC Press LLC, Boca Raton FL, 2017, pp. 233–256.

[ČD] S.Lj. Čukić, E. Delucchi. Simplicial shellable spheres via combinatorial blowups, *Proc. Amer. Math. Soc.* 135 (2007), no. 8, 2403–2414.

[DH] E. Delucchi, L. Hoessly. Fundamental polytopes of metric trees via hyperplane arrangements, arXiv:1612.05534 [math.CO].

[E] G. Ewald. *Combinatorial convexity and algebraic geometry*, volume 168 of Graduate Texts in Mathematics. Springer-Verlag, 1996.

[E-S] G. Ewald, C. Schulz. Non-starshaped spheres. *Arch. Math.*, Vol. 59, 412–416 (1992).

[Far] M. Farber. *Invitation to Topological Robotics*, EMS Zurich Lectures in Advanced Mathematics, Zurich, 2008.

[GaPa] P. Galashin, G. Panina. Simple game induced manifolds. *J. Knot Theory Ramif.* 25(12), (2016). arXiv:1311.6966 [math.GT].

[GP-1] J.E. Goodman and R. Pollack. There are asymptotically far fewer polytopes than we thought. *Bull. Amer. Math. Soc.*, 14 (1986), 127–129.

[GP-2] J.E. Goodman and R. Pollack. Upper bounds for configurations and polytopes in $\mathbb{R}^d$, *Discrete Comput. Geom.*, 1 (1986), 219–227.

[GoPe] J. Gordon, F. Petrov. Combinatorics of the Lipschitz polytope. *Arnold Math. J.* **3** (2), 205–218 (2017). arXiv:1608.06848 [math.CO].

[JJZ] F.D. Jevtić, M. Jelić, and R.T. Živaljević. Cyclohedron and Kantorovich-Rubinstein Polytopes, *Arnold Mathematical Journal*, April 2018, Vol. 4, 87–112.

[JNPZ] D. Jojić, I. Nekrasov, G. Panina, R. Živaljević, Alexander r-tuples and Bier complexes, *Publ. Inst. Math. (Beograd) (N.S.)* 104(118) (2018), 1–22.

[Ka] G. Kalai. Many triangulated spheres, *Discrete Comput. Geom.*, 3 (1988), 1–14.

[K42] L.V. Kantorovich. On the translocation of masses, *Dokl. Akad. Nauk SSSR* 37 (1942), 227–229 (in Russian).

[K-R] L.V. Kantorovich and G.Sh. Rubinshtein. On a space of totally additive functions, *Vestnik Leningrad. Univ.* 13 (1958), no. 7, 52–59 (in Russian).

[Lo1] M. de Longueville. Bier spheres and barycentric subdivision, *J. Comb. Theory Ser. A* 105 (2004), 355–357.

[Lo2] M. de Longueville. *A Course in Topological Combinatorics*, Universitext, Springer, 238 pages, 2013, ISBN 978-1-4419-7909-4.

[M03] J. Matoušek. *Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*. Universitext, Springer-Verlag, Heidelberg, 2003 (Corrected 2nd printing 2008).

[MPV] J. Melleray, F. Petrov, A. Vershik. Linearly rigid metric spaces and the embedding problem. *Fundam. Math.*, 199(2):177–194, 2014.

[NSW] E. Nevo, F. Santos, S. Wilson. Many triangulated odd-dimensional spheres, *Mathematische Annalen*, 364 (2016), 737–762.

[P-Z] J. Pfeifle and G.M. Ziegler. Many triangulated 3-spheres. *Math. Ann.*, 330 (2004), 829–837.

[Tim] M. Timotijević. Note on combinatorial structure of self-dual simplicial complexes, *Mat. Vesnik*, 71:104–122, 2019.

[Ve1] A.M. Vershik. Kantorovich metric: initial history and little-known applications, *Journal of Mathematical Sciences*, 2006, Vol. 133, Issue 4, 1410–1417.

[Ve2] A.M. Vershik. Long history of the Monge-Kantorovich transportation problem. *Math. Intell.*, 32(4):1–9, 2013.

[Ve3] A.M. Vershik. The problem of describing central measures on the path spaces of graded graphs. *Funct. Anal. Appl.*, 48(4):26–46, 2014.

[Ve4] A.M. Vershik. Classification of finite metric spaces and combinatorics of convex polytopes. *Arnold Math. J.*, 1(1):75–81, 2015.

[Vi1] C. Villani. *Topics in Optimal Transportation*, Graduate Studies in Mathematics Vol. 58, Amer. Math. Soc. 2003.

[Vi2] C. Villani. *Optimal Transport: Old and New*, Springer Verlag, Grundlehren der mathematischen Wissenschaften, 2009.