Filomat 32:8 (2018), 3031–3042
https://doi.org/10.2298/FIL1808031G

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Relations between Ordinary and Multiplicative Degree-Based Topological Indices

Ivan Gutman[a], Igor Milovanović[b], Emina Milovanović[b]

[a]Faculty of Science, University of Kragujevac, P. O. Box 60, 34000 Kragujevac, Serbia
[b]Faculty of Electronics Engineering, University of Niš, A. Medvedeva 14, 18000 Niš, Serbia

Abstract. Let G be a simple connected graph with n vertices and m edges, and sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. If vertices i and j are adjacent, we write $i \sim j$. Denote by Π_1, Π_1^*, Q_α and H_α the multiplicative Zagreb index, multiplicative sum Zagreb index, general first Zagreb index, and general sum-connectivity index, respectively. These indices are defined as $\Pi_1 = \prod_{i=1}^n d_i^2$, $\Pi_1^* = \prod_{i\sim j}(d_i + d_j)$, $Q_\alpha = \sum_{i=1}^n d_i^\alpha$ and $H_\alpha = \sum_{i\sim j}(d_i + d_j)^\alpha$. We establish upper and lower bounds for the differences $H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}}$ and $Q_\alpha - n\,(\Pi_1)^{\frac{\alpha}{2n}}$. In this way we generalize a number of results that were earlier reported in the literature.

1. Introduction

Let G be a simple connected graph with vertex set $V = \{1, 2, \dots, n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. Further, let $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$, and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$ be sequences of vertex and edge degrees, respectively. Throughout the paper we will use the following (standard) notation: $\Delta = d_1$, $\Delta_1 = d_2$, $\delta = d_n$, $\delta_1 = d_{n-1}$, $\Delta_{e_1} = d(e_1) + 2$, $\Delta_{e_2} = d(e_2) + 2$, $\delta_{e_1} = d(e_m) + 2$, $\delta_{e_2} = d(e_{m-1}) + 2$. If the vertices i and j are adjacent, we write $i \sim j$. As usual, $L(G)$ denotes a line graph of G.

Two vertex-degree based topological indices, the first and the second Zagreb index, M_1 and M_2, are defined as [19, 22, 23]

$$M_1 = M_1(G) = \sum_{i=1}^n d_i^2 \qquad \text{and} \qquad M_2 = M_2(G) = \sum_{i\sim j} d_i\, d_j\,.$$

For details and further references on these indices see [4, 5, 20, 37].

As shown in [37], the first Zagreb index can be also expressed as

$$M_1 = \sum_{i\sim j}(d_i + d_j)\,. \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 05C12; Secondary 05C50
Keywords. Multiplicative Zagreb index; multiplicative sum Zagreb index; general first Zagreb index; general sum-connectivity index.
Received: 20 July 2017; Accepted: 27 September 2017
Communicated by Dragan S. Djordjević
Research supported by Serbian Ministry of Education, Science and Technological Development, Grant No TR-32009.
Email addresses: gutman@kg.ac.rs (Ivan Gutman), igor@elfak.ni.ac.rs (Igor Milovanović), ema@elfak.ni.ac.rs (Emina Milovanović)

Bearing in mind that for the edge e connecting the vertices i and j,

$$d(e) = d_i + d_j - 2\,,$$

the index M_1 can also be considered as an edge-degree based topological index, since according to (1) holds [32]

$$M_1 = \sum_{i=1}^{m} (d(e_i) + 2)\,.$$

A so-called forgotten topological index, F, is defined as [13] (see also [14]):

$$F = F(G) = \sum_{i=1}^{n} d_i^3\,.$$

By analogy to M_1, the invariant F can be written in the following way [32]

$$F = \sum_{i\sim j} (d_i^2 + d_j^2) = \sum_{i\sim j} (d_i + d_j)^2 - 2M_2\,.$$

The general sum-connectivity index, denoted by H_α, is defined as [51]:

$$H_\alpha = H_\alpha(G) = \sum_{i\sim j} (d_i + d_j)^\alpha\,,$$

where α is an arbitrary real number. It can be easily observed that

$$H_\alpha = \sum_{i=1}^{m} (d(e_i) + 2)^\alpha\,, \qquad H_0 = m\,.$$

Hence, H_α can be considered as edge-degree-based topological index as well. It can be easily verified that $M_1 = H_1$, $\chi = H_{-\frac{1}{2}}$ (sum-connectivity index introduced in [50]), $H = 2H_{-1}$ (harmonic index defined in [11]).

The general first Zagreb index, Q_α, is defined as [29]:

$$Q_\alpha = Q_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha\,,$$

where α is an arbitrary real number. Obviously, $Q_2 = M_1$, $Q_3 = F$, $Q_{-1} = ID$ and $Q_{-1/2} = {}^0R$, where

$$ID = \sum_{i=1}^{n} \frac{1}{d_i}$$

is the inverse degree index [7, 8, 11], whereas

$${}^0R = \sum_{i=1}^{n} \frac{1}{\sqrt{d_i}}$$

is the zeroth–order Randić index [26, 28].

Multiplicative versions of topological indices were proposed in 2010 [40, 41], whereas the first and second multiplicative Zagreb indices, denoted by Π_1 and Π_2, respectively, were first considered in a paper [18] published in 2011, and were promptly followed by numerous additional studies [9, 10, 15, 24, 30, 39, 42, 44, 46, 47]. These indices are defined as:

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2\,, \qquad \Pi_2 = \Pi_2(G) = \prod_{i\sim j} d_i\, d_j\,.$$

One year later, the multiplicative sum-Zagreb index, Π_1^*, was introduced [10], defined as

$$\Pi_1^* = \Pi_1^*(G) = \prod_{i\sim j}(d_i + d_j)\,.$$

Π_1^* can be also be viewed as an edge-degree-based topological index since

$$\Pi_1^*(G) = \prod_{i=1}^{m}(d(e_i) + 2)\,.$$

It should be mentioned that much earlier, the product of vertex degrees was considered by Narumi and Katayama [35, 36], which essentially is the oldest multiplicative Zagreb–type index.

Further details on the multiplicative Zagreb indices can be found in the recent papers [1, 25, 43, 45] and the references quoted therein.

In this paper, we are interested in establishing upper and lower bounds for the differences

$$H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \qquad \text{and} \qquad Q_\alpha - n\,(\Pi_1)^{\frac{\alpha}{2n}}\,.$$

By achieving this goal, we will generalize a number of results that were earlier reported in the literature. In particular, in [39], the following inequalities were shown that:

$$2m - n\,(\Pi_1)^{\frac{1}{2n}} \geq 0\,, \tag{2}$$

$$M_1 - n\,(\Pi_1)^{\frac{1}{n}} \geq 0\,, \tag{3}$$

$$M_2 - m\,(\Pi_2)^{\frac{1}{m}} \geq 0\,. \tag{4}$$

In [44] it was proven that

$$M_1 - m\left(\Pi_1^*\right)^{\frac{1}{m}} \geq 0 \tag{5}$$

whereas in [12] that

$$F + 2M_2 - m\left(\Pi_1^*\right)^{\frac{2}{m}} \geq 0\,. \tag{6}$$

2. Preliminaries

In this section we recall some analytical inequalities for real number sequences that will be used in the subsequent considerations.

Let $a_i = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, p$, be positive real number sequences with the properties

$$0 < r_1 \leq a_i \leq R_1 < +\infty \qquad \text{and} \qquad 0 < r_2 \leq b_i \leq R_2 < +\infty\,.$$

In [2] (see also [33]) the following inequality was proven

$$\left| p\sum_{i=1}^{p} a_i b_i - \sum_{i=1}^{p} a_i \sum_{i=1}^{p} b_i \right| \leq p^2\gamma(p)(R_1 - r_1)(R_2 - r_2)\,, \tag{7}$$

where

$$\gamma(p) = \frac{1}{p}\left\lfloor\frac{p}{2}\right\rfloor\left(1 - \frac{1}{p}\left\lfloor\frac{p}{2}\right\rfloor\right) = \frac{1}{4}\left(1 - \frac{(-1)^{p+1}+1}{2p^2}\right).$$

For the positive real number sequence $a = (a_i)$, $i = 1, 2, \ldots, p$, the following inequality was proven in [48] (see also [27])

$$\left(\sum_{i=1}^{p} \sqrt{a_i} \right)^2 \leq (p-1) \sum_{i=1}^{p} a_i + p \left(\prod_{i=1}^{p} a_i \right)^{1/p} . \tag{8}$$

For the sequence of positive real numbers $a = (a_i)$, $i = 1, 2, \ldots, p$, with the property $a_1 \geq a_2 \geq \cdots \geq a_p > 0$, in [6] the following was proven

$$\sum_{i=1}^{p} a_i - p \left(\prod_{i=1}^{p} a_i \right)^{1/p} \geq \left(\sqrt{a_1} - \sqrt{a_p} \right)^2 . \tag{9}$$

Before we proceed, let us define one special class of d-regular graphs Γ_d (see [38]). Let $N(i)$ be a set of all neighbors of the vertex i, i.e., $N(i) = \{k \,|\, k \in V,\ k \sim i\}$. Let $d(i, j)$ be the distance between the vertices i and j. Denote by Γ_d a set of all d-regular graphs, $1 \leq d \leq n-1$, with diameter 2, and $|N(i) \cap N(j)| = d$ for $i \nsim j$.

3. Main results

In the next theorem, we establish upper and lower bounds for the difference $Q_\alpha - n(\Pi_1)^{\alpha/2n}$, in terms of the number of vertices and minimal and maximal vertex degrees.

Theorem 3.1. *Let G be a simple connected graph with $n \geq 2$ vertices. Then, for any real $\alpha \geq 0$,*

$$\left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}} \right)^2 \leq Q_\alpha - n \left(\Pi_1 \right)^{\frac{\alpha}{2n}} \leq n^2 \gamma(n) \left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}} \right)^2 . \tag{10}$$

If $\alpha \leq 0$, then

$$\left(\delta^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}} \right)^2 \leq Q_\alpha - n \left(\Pi_1 \right)^{\frac{\alpha}{2n}} \leq n^2 \gamma(n) \left(\delta^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}} \right)^2 . \tag{11}$$

Equalities on the right–hand sides hold if and only if G is regular. Equalities on the left–hand sides hold if and only if $d_2 = \cdots = d_{n-1} = \sqrt{d_1 \, d_n}$.

Proof. For $p = n$, $a_i = b_i = d_i^{\frac{\alpha}{2}}$, $R_1 = R_2 = \Delta^{\frac{\alpha}{2}}$, $r_1 = r_2 = \delta^{\frac{\alpha}{2}}$, $\alpha \geq 0$, $i = 1, 2, \ldots, n$, the inequality (7) becomes

$$n \sum_{i=1}^{n} d_i^{\alpha} - \left(\sum_{i=1}^{n} d_i^{\frac{\alpha}{2}} \right)^2 \leq n^2 \gamma(n) \left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}} \right)^2 ,$$

i.e.,

$$n Q_\alpha - \left(\sum_{i=1}^{n} d_i^{\frac{\alpha}{2}} \right)^2 \leq n^2 \gamma(n) \left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}} \right)^2 . \tag{12}$$

For $p = n$, $\alpha \geq 0$, $a_i = d_i^{\alpha}$, $i = 1, 2, \ldots, n$, the inequality (8) transforms into

$$\left(\sum_{i=1}^{n} d_i^{\frac{\alpha}{2}} \right)^2 \leq (n-1) \sum_{i=1}^{n} d_i^{\alpha} + n \left(\prod_{i=1}^{n} d_i^{\alpha} \right)^{1/n} ,$$

i.e.,

$$\left(\sum_{i=1}^{n} d_I^{\frac{\alpha}{2}} \right)^2 \leq (n-1) Q_\alpha + n \left(\Pi_1 \right)^{\frac{\alpha}{2n}} . \tag{13}$$

From (12) and (13) the inequality (10) is obtained.

Equality in (13) holds if and only if $d_1 = \cdots = d_n$, so the equality on the right–hand side of (10) holds if and only if G is regular.

For $p = n$, $\alpha \geq 0$, $a_i = d_i^\alpha$, $i = 1, 2, \cdots, n$, the inequality (9) becomes

$$\sum_{i=1}^{n} d_i^\alpha - n\left(\prod_{i=1}^{n} d_i^\alpha\right)^{1/n} \geq \left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2,$$

i.e.,

$$Q_\alpha - n\,(\Pi_1)^{\frac{\alpha}{2n}} \geq \left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2, \tag{14}$$

which coincides with the left–hand side of (10).

Equality in (14) holds if and only if $d_2 = \cdots = d_{n-1} = \sqrt{d_1\, d_n}$. Equality on the left–hand side of (10) holds under same condition.

Inequalities (14) can be verified in an analogous manner. □

In a similar way, we arrive at the following:

Theorem 3.2. *Let G be a simple connected graph with n vertices. If $n \geq 3$ and $\alpha \geq 0$, then*

$$\begin{aligned}\Delta^\alpha + \left(\Delta_1^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2 &\leq Q_\alpha - (n-1)\left(\frac{\Pi_1}{\Delta^2}\right)^{\frac{\alpha}{2(n-1)}} \\ &\leq \Delta^\alpha + (n-1)^2\gamma(n-1)\left(\Delta_1^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2.\end{aligned}$$

If $n \geq 3$ and $\alpha \leq 0$, then

$$\begin{aligned}\Delta^\alpha + \left(\delta^{\frac{\alpha}{2}} - \Delta_1^{\frac{\alpha}{2}}\right)^2 &\leq Q_\alpha - (n-1)\left(\frac{\Pi_1}{\Delta^2}\right)^{\frac{\alpha}{2(n-1)}} \\ &\leq \Delta^\alpha + (n-1)^2\gamma(n-1)\left(\delta^{\frac{\alpha}{2}} - \Delta_1^{\frac{\alpha}{2}}\right)^2.\end{aligned}$$

Equalities on the right–hand sides hold if and only if $\Delta_1 = d_2 = \cdots = d_n = \delta$. Equalities on the left–hand sides hold if and only if $d_3 = \cdots = d_{n-1} = \sqrt{\Delta_1 \delta}$.

Theorem 3.3. *Let G be a simple connected graph with n vertices. If $n \geq 3$ and $\alpha \geq 0$, then*

$$\delta^{\frac{\alpha}{2}} + \left(\Delta^{\frac{\alpha}{2}} - \delta_1^{\frac{\alpha}{2}}\right)^2 \leq Q_\alpha - (n-1)\left(\frac{\Pi_1}{\delta^2}\right)^{\frac{\alpha}{2(n-1)}} \leq \delta^\alpha + (n-1)^2\gamma(n-1)\left(\Delta^{\frac{\alpha}{2}} - \delta_1^{\frac{\alpha}{2}}\right)^2.$$

If $n \geq 3$ and $\alpha \leq 0$, then

$$\delta^{\frac{\alpha}{2}} + \left(\delta_1^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}}\right)^2 \leq Q_\alpha - (n-1)\left(\frac{\Pi_1}{\delta^2}\right)^{\frac{\alpha}{2(n-1)}} \leq \delta^\alpha + (n-1)^2\gamma(n-1)\left(\delta_1^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}}\right)^2.$$

Equalities on the right–hand side of the above inequalities hold if and only if $\Delta = d_1 = \cdots = d_{n-1} = \delta_1$, and on the left–hand side if and only if $\Delta_1 = d_2 = \cdots = d_{n-2} = \sqrt{\Delta\, \delta_1}$.

Theorem 3.4. *Let G be a simple connected graph with n vertices. If $n \geq 4$ and $\alpha \geq 0$, then*

$$\begin{aligned}\Delta^\alpha + \delta^\alpha + \left(\Delta_1^{\frac{\alpha}{2}} - \delta_1^{\frac{\alpha}{2}}\right)^2 &\leq Q_\alpha - (n-2)\left(\frac{\Pi_1}{\Delta^2\,\delta^2}\right)^{\frac{\alpha}{2(n-1)}} \\ &\leq \Delta^\alpha + \delta^\alpha + (n-2)^2\gamma(n-2)\left(\Delta_1^{\frac{\alpha}{2}} - \delta_1^{\frac{\alpha}{2}}\right)^2.\end{aligned}$$

If $n \geq 4$ and $\alpha \leq 0$, then

$$\begin{aligned} \Delta^{\alpha} + \delta^{\alpha} + \left(\delta_1^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}}\right)^2 &\leq Q_{\alpha} - (n-2)\left(\frac{\Pi_1}{\Delta^2\,\delta^2}\right)^{\frac{\alpha}{2(n-1)}} \\ &\leq \Delta^{\alpha} + \delta^{\alpha} + (n-2)\gamma(n-2)\left(\delta_1^{\frac{\alpha}{2}} - \Delta_1^{\frac{\alpha}{2}}\right)^2 . \end{aligned}$$

Equalities on the left–hand sides of the above inequalities hold if and only if $\Delta_1 = d_2 = \cdots = d_{n-1} = \delta_1$, and on the right–hand sides if and only if $d_3 = \cdots = d_{n-2} = \sqrt{\Delta_1\,\delta_1}$.

In the next corollary we point out some inequalities that are obtained from (10) and (11) for some particular values of the parameter α.

Corollary 3.5. *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$\frac{\left(\sqrt[4]{\Delta} - \sqrt[4]{\delta}\right)^2}{\sqrt{\Delta\delta}} \leq {}^0R - n\,(\Pi_1)^{-\frac{1}{4n}} \leq n^2\gamma(n)\frac{\left(\sqrt[4]{\Delta} - \sqrt[4]{\delta}\right)^2}{\sqrt{\Delta\delta}}\,,$$

$$\frac{\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2}{\Delta\delta} \leq ID - n\,(\Pi_1)^{-\frac{1}{2n}} \leq n^2\gamma(n)\frac{\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2}{\Delta\delta}\,, \tag{15}$$

$$\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2 \leq 2m - n\,(\Pi_1)^{\frac{1}{2n}} \leq n^2\gamma(n)\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2\,, \tag{16}$$

$$(\Delta - \delta)^2 \leq M_1 - n\,(\Pi_1)^{\frac{1}{n}} \leq n^2\gamma(n)\,(\Delta - \delta)^2\,, \tag{17}$$

$$\left(\Delta^{\frac{3}{2}} - \delta^{\frac{3}{2}}\right)^2 \leq F - n\,(\Pi_1)^{\frac{3}{2n}} \leq n^2\gamma(n)\left(\Delta^{\frac{3}{2}} - \delta^{\frac{3}{2}}\right)^2 . \tag{18}$$

Remark 3.6. *The left–hand side inequalities in* (16) *and* (17) *are stronger than* (2) *and* (3)*, respectively.*

Since $2R_{-1} \leq ID$ (see [31]), where $R_{-1} = \sum_{i\sim j}\frac{1}{d_i\,d_j}$ is an often used Randić–type index [3, 28], the following corollary of Theorem 3.1 is valid:

Corollary 3.7. *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$2R_{-1} - n\,(\Pi_1)^{-\frac{1}{2n}} \leq n^2\gamma(n)\frac{(\sqrt{\Delta} - \sqrt{\delta})^2}{\Delta\,\delta}\,,$$

with equality if and only if G is regular.

Since $F \geq 2M_2$, based on the right part of (18) the following result is obtained.

Corollary 3.8. *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$2M_2 - n\,(\Pi_1)^{\frac{3}{2n}} \leq n^2\gamma(n)\left(\Delta^{\frac{3}{2}} - \delta^{\frac{3}{2}}\right)^2,$$

with equality if and only if G is regular.

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ be the Laplacian eigenvalues values of the graph G [16, 17, 34]. Then the Kirchhoff index, Kf, is defined as [21] (see also [52])

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}\,.$$

Corollary 3.9. *Let G be a simple connected graph with n vertices. If $n \geq 2$ then*

$$Kf(G) \geq -1 + (n-1)\left(n\,(\Pi_1)^{-\frac{1}{2n}} + \frac{\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2}{\Delta\,\delta} \right). \tag{19}$$

If $n \geq 3$, then

$$Kf(G) \geq \frac{n-1-\Delta}{\Delta} + (n-1)\left((n-1)\left(\frac{\Pi_1}{\Delta^2}\right)^{-\frac{1}{2(n-1)}} + \frac{\left(\sqrt{\Delta_1} - \sqrt{\delta}\right)^2}{\Delta_1\,\delta} \right). \tag{20}$$

Equality in (19) *holds if and only if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ when n is even, or $G \in \Gamma_d$. Equality in* (20) *holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ for even n, or $G \cong K_{1,n-1}$, or $G \in \Gamma_d$.*

Proof. In [49], the following inequality for the Kirchhoff index was reported:

$$Kf(G) \geq -1 + (n-1)\sum_{i=1}^{n}\frac{1}{d} = -1 + (n-1)ID\,. \tag{21}$$

The inequality (19) is obtained from (21) and the left part of (15).

For $\alpha = -1$, from Theorem 3.2 the following is obtained:

$$ID - (n-1)\left(\frac{\Pi_1}{\Delta^2}\right)^{-\frac{1}{2(n-1)}} \geq \frac{1}{\Delta} + \frac{\left(\sqrt{\Delta_1} - \sqrt{\delta}\right)^2}{\Delta_1\,\delta}\,.$$

According to the above and inequality (21), inequality (20) is obtained. □

In the next theorem we establish lower and upper bounds for the difference $H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}}$ depending on the parameters m, Δ_{e_1}, and δ_{e_1}.

Theorem 3.10. *Let G be a simple graph with $m \geq 1$ edges. If $\alpha \geq 0$ then*

$$\left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}}\right)^2 \leq H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq m^2\gamma(m)\left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}}\right)^2. \tag{22}$$

If $\alpha \leq 0$, then

$$\left(\delta_{e_1}^{\frac{\alpha}{2}} - \Delta_{e_1}^{\frac{\alpha}{2}}\right)^2 \leq H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq m^2\gamma(m)\left(\delta_{e_1}^{\frac{\alpha}{2}} - \Delta_{e_1}^{\frac{\alpha}{2}}\right)^2.$$

Equalities on the right–hand sides of the above inequalities are attained if and only if $L(G)$ is regular. Equalities on the left–hand sides hold if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2} = \sqrt{\Delta_{e_1}\delta_{e_1}}$.

Proof. For $p = m$, $\alpha \geq 0$, $a_i = b_i = (d(e_i) + 2)^{\frac{\alpha}{2}}$, $R_1 = R_2 = \Delta_{e_1}^{\frac{\alpha}{2}}$, $r_1 = r_2 = \delta_{e_1}^{\frac{\alpha}{2}}$, $i = 1, 2, \ldots, m$, the inequality (7) becomes

$$m\sum_{i=1}^{m}(d(e_i) + 2)^{\alpha} - \left(\sum_{i=1}^{m}(d(e_i) + 2)^{\frac{\alpha}{2}}\right)^2 \leq m^2\gamma(m)\left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}}\right)^2,$$

i.e.,

$$mH_\alpha - \left(\sum_{i=1}^{m}(d(e_i) + 2)^{\frac{\alpha}{2}}\right)^2 \leq m^2\gamma(m)\left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}}\right)^2. \tag{23}$$

For $p = m$, $\alpha \geq o$, $a_i = (d(e_i) + 2)^\alpha$, $i = 1, 2, \ldots, m$, the inequality (8) transforms into

$$\left(\sum_{i=1}^{m} (d(e_i) + 2)^{\frac{\alpha}{2}} \right)^2 \leq (m-1) \sum_{i=1}^{m} (d(e_i) + 2)^\alpha + m \left(\prod_{i=1}^{m} (d(e_i) + 2)^\alpha \right)^{\frac{1}{m}} ,$$

i.e.,

$$\left(\sum_{i=1}^{m} (d(e_i) + 2)^{\frac{\alpha}{2}} \right)^2 \leq (m-1) H_\alpha + m \left(\Pi_1^* \right)^{\frac{\alpha}{m}} . \tag{24}$$

The right–hand side of (22) is obtained from (23) and (24), .

Equality in (24) holds if and only if $\Delta_{e_1} = d(e_1) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$. Therefore, equality on the right–hand side of (22) holds if and only if $L(G)$ is regular.

For $p = m$, $\alpha \geq 0$, $a_i = (d(e_i) + 2)^\alpha$, $a_1 = \Delta_{e_1}^\alpha$, $a_m = \delta_{e_1}^\alpha$, $i = 1, 2, \ldots, m$, the inequality (9) becomes

$$\sum_{i=1}^{m} (d(e_i) + 2)^\alpha - m \left(\prod_{i=1}^{m} (d(e_i) + 2)^\alpha \right)^{\frac{1}{m}} \geq \left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}} \right)^2 ,$$

i.e.,

$$H_\alpha - m \left(\Pi_1^* \right)^{\frac{\alpha}{m}} \geq \left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}} \right)^2 ,$$

which is just the left–hand side of (22). Equality in the above inequality, and therefore on the left–hand side of (22), holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-2}) + 2 = \delta_{e_2} = \sqrt{\Delta_{e_1} \delta_{e_1}}$.

For the case $\alpha \leq 0$ the inequalities are proved in a similar way. □

The same procedure as in the case of Theorem 3.10 can be applied to deduce the following result.

Theorem 3.11. *Let G be a simple connected graph with m edges. If $m \geq 2$ and $\alpha \geq 0$, then*

$$\begin{aligned} \Delta_{e_1}^\alpha + \left(\Delta_{e_2}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}} \right)^2 &\leq H_\alpha - (m-1) \left(\frac{\Pi_1^*}{\Delta_{e_1}} \right)^{\frac{\alpha}{m-1}} \\ &\leq \Delta_{e_1}^\alpha + (m-1)^2 \gamma(m-1) \left(\Delta_{e_2}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}} \right)^2 . \end{aligned}$$

If $m \geq 2$ and $\alpha \leq 0$, then

$$\begin{aligned} \Delta_{e_1}^\alpha + \left(\delta_{e_1}^{\frac{\alpha}{2}} - \Delta_{e_2}^{\frac{\alpha}{2}} \right)^2 &\leq H_\alpha - (m-1) \left(\frac{\Pi_1^*}{\Delta_{e_1}} \right)^{\frac{\alpha}{m-1}} \\ &\leq \Delta_{e_1}^\alpha + (m-1)^2 \gamma(m-1) \left(\delta_{e_1}^{\frac{\alpha}{2}} - \Delta_{e_2}^{\frac{\alpha}{2}} \right)^2 . \end{aligned}$$

Equalities on the right–hand sides hold if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_1}$, and on the left–hand sides if and only if $d(e_3) + 2 = \cdots = d(e_m) + 2 = \delta_{e_2} = \sqrt{\Delta_{e_2} \delta_{e_1}}$.

Theorem 3.12. *Let G be a simple connected graph with m edges. If $m \geq 3$ and $\alpha \geq 0$, then*

$$\begin{aligned} \Delta_{e_1}^\alpha + \delta_{e_1}^\alpha + \left(\Delta_{e_2}^{\frac{\alpha}{2}} - \delta_{e_2}^{\frac{\alpha}{2}} \right)^2 &\leq H_\alpha - (m-2) \left(\frac{\Pi_1^*}{\Delta_{e_1} \delta_{e_1}} \right)^{\frac{\alpha}{m-2}} \\ &\leq \Delta_{e_1}^\alpha + \delta_{e_1}^\alpha + (m-2)^2 \gamma(m-2) \left(\Delta_{e_2}^{\frac{\alpha}{2}} - \delta_{e_2}^{\frac{\alpha}{2}} \right)^2 . \end{aligned}$$

If $m \geq 3$ and $\alpha \leq 0$, then

$$\begin{aligned} \Delta_{e_1}^{\alpha} + \delta_{e_1}^{\alpha} + \left(\delta_{e_2}^{\frac{\alpha}{2}} - \Delta_{e_2}^{\frac{\alpha}{2}}\right)^2 &\leq H_\alpha - (m-2)\left(\frac{\Pi_1^*}{\Delta_{e_1}\delta_{e_1}}\right)^{\frac{\alpha}{m-2}} \\ &\leq \Delta_{e_1}^{\alpha} + \delta_{e_1}^{\alpha} + (m-2)^2\gamma(m-2)\left(\delta_{e_2}^{\frac{\alpha}{2}} - \Delta_{e_2}^{\frac{\alpha}{2}}\right)^2 . \end{aligned}$$

Equalities on the right–hand sides hold if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$, and on the left–hand sides if and only if $d(e_3) + 2 = \cdots = d(e_{m-2}) + 2 = \sqrt{\Delta_{e_2}\delta_{e_2}}$.

Since $2\delta \leq \delta_{e_1} \leq \Delta_{e_1} \leq 2\Delta$, the following corollary of Theorem 3.10 is valid.

Corollary 3.13. *Let G be a simple connected graph with $m \geq 1$ edges. If $\alpha \geq 0$, then*

$$H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq 2^\alpha m^2\gamma(m)\left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2 .$$

If $\alpha \leq 0$, then

$$H_\alpha - m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq 2^\alpha m^2\gamma(m)\left(\delta^{\frac{\alpha}{2}} - \Delta^{\frac{\alpha}{2}}\right)^2 .$$

In both cases equalities hold if and only if G is regular.

We now state some inequalities resulting from Theorem 3.10 and Corollary 3.13, pertaining to particular values of the parameter α, namely for $\alpha = -\frac{1}{2}$, $\alpha = -1$, $\alpha = 1$, and $\alpha = 2$, respectively.

Corollary 3.14. *Let G be a simple connected graph with $m \geq$ edges. Then*

$$\begin{aligned} \frac{\left(\sqrt[4]{\Delta_{e_1}} - \sqrt[4]{\delta_{e_1}}\right)^2}{\sqrt{\Delta_{e_1}\delta_{e_1}}} \leq \chi - m\left(\Pi_1^*\right)^{-\frac{1}{2m}} &\leq m^2\gamma(m)\frac{\left(\sqrt[4]{\Delta_{e_1}} - \sqrt[4]{\delta_{e_1}}\right)^2}{\sqrt{\Delta_{e_1}\delta_{e_1}}} \\ &\leq m^2\gamma(m)\frac{\left(\sqrt[4]{\Delta} - \sqrt[4]{\delta}\right)^2}{\sqrt{2\Delta\,\delta}} , \end{aligned}$$

$$\begin{aligned} \frac{\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2}{\Delta_{e_1}\delta_{e_1}} \leq \frac{1}{2}H - m\left(\Pi_1^*\right)^{-\frac{1}{m}} &\leq m^2\gamma(m)\frac{\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2}{\Delta_{e_1}\delta_{e_1}} \\ &\leq m^2\gamma(m)\frac{\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2}{2\Delta\,\delta} , \end{aligned}$$

$$\begin{aligned} \left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2 \leq M_1 - m\left(\Pi_1^*\right)^{\frac{1}{m}} &\leq m^2\gamma(m)\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2 \\ &\leq 2m^2\gamma(m)\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2 , \end{aligned} \tag{25}$$

$$\begin{aligned} (\Delta_{e_1} - \delta_{e_1})^2 - 2M_2 \leq F - m\left(\Pi_1^*\right)^{\frac{2}{m}} &\leq m^2\gamma(m)\left(\Delta_{e_1} - \delta_{e_1}\right)^2 - 2M_2 \\ &\leq 4m^2\gamma(m)(\Delta - \delta)^2 - 2M_2 . \end{aligned} \tag{26}$$

Remark 3.15. *Left inequality of* (25) *is stronger than* (5)*, and left inequality of* (26) *is stronger than* (6).

As $F \geq 2M_2$, from (26) we obtain:

Corollary 3.16. *Let G be a simple connected graph with $m \geq 1$ edges. Then*

$$2F - m\left(\Pi_1^*\right)^{\frac{2}{m}} \geq \left(\Delta_{e_1} - \delta_{e_1}\right)^2 ,$$

$$4M_2 - m\left(\Pi_1^*\right)^{\frac{2}{m}} \leq m^2\gamma(m)\left(\Delta_{e_1} - \delta_{e_1}\right)^2 \leq 4m^2\gamma(m)(\Delta - \delta)^2 .$$

Equalities hold if and only G is regular.

In the next theorem we establish a relationship between H_α and Π_2.

Theorem 3.17. *Let G be a simple connected graph with n vertices and $m \geq 1$ edges. Then for any $\alpha \geq 0$*

$$H_\alpha - \frac{n^\alpha}{m^{\alpha-1}}\left(\Pi_2\right)^{\frac{\alpha}{m}} \leq m^2\gamma(m)\left(\Delta_{e_1}^{\frac{\alpha}{2}} - \delta_{e_1}^{\frac{\alpha}{2}}\right)^2 \leq 2^\alpha m^2\gamma(m)\left(\Delta^{\frac{\alpha}{2}} - \delta^{\frac{\alpha}{2}}\right)^2 . \tag{27}$$

Equality on the left–hand side of (27) *holds if and only if $L(G)$ is regular, and on the right–hand side if and only if G is regular.*

Proof. According to

$$n = \sum_{i\sim j}\frac{d_i + d_j}{d_id_j} \geq m\left(\prod_{i\sim j}\frac{d_i + d_j}{d_id_j}\right)^{\frac{1}{m}} = m\frac{\left(\Pi_1^*\right)^{\frac{1}{m}}}{\left(\Pi_2\right)^{\frac{1}{m}}} ,$$

we have that

$$m\left(\Pi_1^*\right)^{\frac{1}{m}} \leq n\left(\Pi_2\right)^{\frac{1}{m}} . \tag{28}$$

If $\alpha \geq 0$ is an arbitrary real number, then

$$m^\alpha\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq n^\alpha\left(\Pi_2\right)^{\frac{\alpha}{m}} ,$$

i.e.,

$$m\left(\Pi_1^*\right)^{\frac{\alpha}{m}} \leq \frac{n^\alpha}{m^{\alpha-1}}\left(\Pi_2\right)^{\frac{\alpha}{m}} .$$

From the above and the right–hand side of (22), the left-hand side of inequality (27) follows. □

Corollary 3.18. *Let G be a simple connected graph with n vertices and $m \geq 1$ edges. Then*

$$M_1 - n\left(\Pi_2\right)^{\frac{1}{m}} \leq m^2\gamma(m)\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2 \leq 2m^2\gamma(m)\left(\sqrt{\Delta} - \sqrt{\delta}\right)^2 ,$$

$$F - \frac{n^2}{m}\left(\Pi_2\right)^{\frac{2}{m}} \leq m^2\gamma(m)\left(\Delta_{e_1} - \delta_{e_1}\right)^2 - 2M_2 \leq 4m^2\gamma(m)(\Delta - \delta)^2 - 2M_2 ,$$

$$4M_2 - \frac{n^2}{m}\left(\Pi_2\right)^{\frac{2}{m}} \leq m^2\gamma(m)\left(\Delta_{e_1} - \delta_{e_1}\right)^2 \leq 4m^2\gamma(m)(\Delta - \delta)^2 .$$

Equalities on the first right–hand sides of the above inequalities are attained if and only if G is regular or biregular. Equalities on the second right–hand sides are attained if and only if G is regular.

In a similar manner as in the case of Theorem 3.17, the following result can be proven.

Theorem 3.19. *Let G be a simple connected graph with n vertices and $m \geq 1$ edges. Then for any real $\alpha \leq 0$*

$$H_\alpha - \frac{n^\alpha}{m^{\alpha-1}}\left(\Pi_2\right)^{\frac{\alpha}{m}} \geq \left(\delta_{e_1}^{\frac{\alpha}{2}} - \Delta_{e_1}^{\frac{\alpha}{2}}\right)^2 . \tag{29}$$

Equality holds if and only if G is a regular or a biregular graph.

For $\alpha = -\frac{1}{2}$ and $\alpha = -1$, we have the following special cases of Theorem 3.19.

Corollary 3.20. *Let G be a simple connected graph with n vertices and $m \geq 1$ edges. Then*

$$\chi - \frac{m\sqrt{m}}{\sqrt{n}} (\Pi_2)^{-\frac{1}{2m}} \geq \frac{\left(\sqrt[4]{\Delta_{e_1}} - \sqrt[4]{\delta_{e_1}}\right)^2}{\sqrt{\Delta_{e_1}\delta_{e_1}}},$$

$$\frac{1}{2}H - \frac{m^2}{n} (\Pi_2)^{-\frac{1}{m}} \geq \frac{\left(\sqrt{\Delta_{e_1}} - \sqrt{\delta_{e_1}}\right)^2}{\Delta_{e_1}\delta_{e_1}}.$$

Remark 3.21. *It can be easily verified that according to* (4) *and* (28), *the following lower bound*

$$M_2 \geq \frac{m^2}{n} \left(\Pi_1^*\right)^{\frac{1}{m}}$$

holds for the second Zagreb index M_2.

References

[1] M. Azari, A. Iranmanesh, Bounds on multiplicative Zagreb indices of graph operations and subdivision operators, In: Bounds in Chemical Graph Theory – Advances (I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović, eds.), Univ. Kragujevac, Kragujevac, 2017, pp. 187–215.
[2] M. Biernacki, H. Pidek, C. Ryll–Nardzewski, Sur une inequality des integralles definies, Univ. Marie Curie–Sklodowska, A4 (1950) 1–4.
[3] B. Bollobás, P. Erdős, Graphs of extremal weights, Ars Comb. 50 (1998) 225–233.
[4] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, Bounds for Zagreb indices, MATCH Commun. Math. Comput. Chem. 78 (2017) 17–100.
[5] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, Zagreb indices: Bounds and Extremal graphs, In: Bounds in Chemical Graph Theory – Basics (I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović. eds.), Univ. Kragujevac, Kragujevac, 2017, pp. 67–153.
[6] V. Cirtoaje, The best lower bound depended on two fixed variables for Jensen's ineauality with order variables, J. Ineq. Appl. 2010 (2010) #12858.
[7] P. Dankelmann, A. Hellwig, L. Volkmann, Inverse degree and edge–connectivity, Discr. Math. 309 (2008) 2943–2947.
[8] P. Dankelmann, H. C. Swart, P. van den Berg, Diameter and inverse degree, Discr. Math. 308 (2008) 670–673.
[9] M. Eliasi, A simple approach to order the multiplicative Zagreb indices of connected graphs, Trans. Comb. 1(4) (2012) 17–24.
[10] M. Eliasi, A. Iranmanesh, I. Gutman, Multiplicative versions of first Zagreb index, MATCH Commun. Math. Comput. Chem. 68 (2012) 217–230.
[11] S. Fajtlowicz, On conjectures of Graffiti II, Congr. Numer. 60 (1987) 187–197.
[12] F. Falati–Nezhad, M. Azari, Bounds of the hyper–Zagreb index, J. Appl. Math. Infor. 34 (3-4) (2016) 319–330.
[13] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem. 53 (2015) 1184–1190.
[14] B. Furtula, I. Gutman, Ž. Kovijanić Vukićević, G. Lekishvili, G. Popivoda, On an old/new degree–based topological index, Bull. Acad. Serbe. Sci. Arts. (Cl. Sci. Math. Natur.) 148 (2015) 19–31.
[15] M. Ghorbani, N. Azimi, Note on multiple Zagreb indices, Iran. J. Math. Chem. 3 (2012) 137–143.
[16] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discr. Math. 7 (1994) 221–229.
[17] R. Grone, R. Merris, V. S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.
[18] I. Gutman, Multiplicative Zagreb indices of trees, Bull. Int. Math. Virt. Inst. 1 (2011) 13–19.
[19] I. Gutman, On the origin of two degree-based topological indices, Bull. Acad. Serbe. Sci. Arts. (Cl. Sci. Math. Natur.) 146 (2014) 39–52.
[20] I. Gutman, K. C. Das, The first Zagreb index 30 years after, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.
[21] I. Gutman, B. Mohar, The quasi-Wiener index and the Kirchhoff indices coincide, J. Chem. Inf. Comput. Sci. 36 (1996) 982–985.
[22] I. Gutman, B. Ruščić, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes. J. Chem. Phys. 62 (1975) 3399–3405.
[23] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons. Chem. Phys. Lett. 17 (1972) 535–538.
[24] A. Iranmanesh, M. A. Hosseinzadeh, I. Gutman, On multiplicative Zagreb indices of graphs, Iran. J. Math. Chem. 3(2) (2012) 145–154.
[25] R. Kazemi, On the multiplicative Zagreb indices of bucket recursive trees, Iran. J. Math. Chem. 8(1) (2017) 37–45.
[26] L. B. Kier, L. H. Hall, The nature of structure–activity relationships and their relation to molecular connectivity, Europ. J. Med. Chem. 12 (1977) 307–312.
[27] H. Kober, On the aritmetic and geometric means and on Hölder's inequality, Proc. Am. Math. Soc. 9 (1958) 452–459.

[28] X. Li, I. Gutman, Mathematical Aspects of Randić–Type Molecular Structure Descriptors, Univ. Kragujevac, Kragujevac, 2006.
[29] X. Li, H. Zhao, Trees with the first the smallest and largest generalized topological indices, MATCH Commun. Math. Comput. Chem. 50 (2004) 57–62.
[30] J. Liu, Q. Zhang, Sharp upper bounds for multiplicative Zagreb indices, MATCH Commun. Math. Comput. Chem. 68 (2012) 231–240.
[31] M. Lu, H. Liu, F. Tian, The connectivity index, MATCH Commun. Math. Comput. Chem. 51 (2004) 149–154.
[32] I. Ž. Milovanović, E. I. Milovanović, I. Gutman, B. Furtula, Some inequalities for the forgotten topological index, Int. J. Appl. Graph Theory 1 (2017) 1–15.
[33] D. S. Mitrinović, P. M. Vasić, Analytic Inequalities, Springer, Berlin, 1970.
[34] B. Mohar, The Laplacian spectrum of graphs, In: Graph Theory, Combinatorics, and Applications (Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk, eds.), Wiley, New York, 1991, pp. 871–898.
[35] H. Narumi, New topological indices for finite and infinite systems, MATCH Commun. Math. Chem. 22 (1987) 195–207.
[36] H. Narumi, M. Katayama, Simple topological index. A newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons, Mem. Fac. Engin. Hokkaido Univ. 16 (1984) 209–214.
[37] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, The Zagreb indices 30 years after, Croat. Chem. Acta 76 (2003) 113–124.
[38] J. L. Palacios, Some additional bounds for the Kirchhoff index, MATCH Commun. Math. Comput. Chem. 75 (2016) 365–372.
[39] T. Réti, I. Gutman, Relations between ordinary and multiplicative Zagreb indices, Bull. Int. Math. Virt. Inst. 2 (2012) 133–140.
[40] R. Todeschini, D. Ballabio, V. Consonni, Novel molecular descriptors based on functions of new vertex degrees, In: Novel Molecular Structure Descriptors - Theory and Applications (I. Gutman, B. Furtula, eds.), Univ. Kragujevac, Kragujevac, 2010, pp. 73–100.
[41] R. Todeschini, V. Consonni, New local vertex invariants and molecular descriptors based on functions of the vertex degrees, MATCH Commun. Math. Comput. Chem. 64 (2010) 359–372.
[42] B. Wang, F. Xia, Narumi–Katayama index of fully loaded unicyclic graphs, South Asian J. Math. 2 (2012) 417–422.
[43] C. Wang, J. B. Liu, S. Wang, Sharp upper bounds for multiplicative Zagreb indices of bipartite graphs with given diameter, Discr. Appl. Math. 227 (2017) 156–165.
[44] H. Wang, H. Bao, A note on multiplicative sum Zagreb index, South Asian J. Math. 2 (2012) 578–583.
[45] S. Wang, C. Wang, L. Chen, J. B. Liu, On extremal multiplicative Zagreb indices of trees with given number of vertices of maximum degree, Discr. Appl. Math. 227 (2017) 166–173.
[46] K. Xu, K. C. Das, Trees, unicyclic and bicyclic graphs extremal with respect to multiplicative sum Zagreb index, MATCH Commun. Math. Comput. Chem. 68 (2012) 257–272.
[47] K. Xu, H. Hua, A unified approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs, MATCH Commun. Math. Comput. Chem. 68 (2012) 241–256.
[48] B. Zhou, I. Gutman, T. Aleksić, A note on the Laplacian energy of graphs, MATCH Commun. Math. Comput. Chem. 60 (2008) 441–446.
[49] B. Zhou, N. Trinajstić, A note on Kirchhoff index, Chem. Phys. Lett. 455 (2008) 120–123.
[50] B. Zhou, N. Trinajstić, On a novel connectivity index, J. Math. Chem. 46 (2009) 1252–1270.
[51] B. Zhou, N. Trinajstić, On general sum-connectivity index, J. Math. Chem. 47 (2010) 210–218.
[52] H. Y. Zhu, D. J. Klein, I. Lukovits, Extensions of the Wiener number, J. Chem. Inf. Comput. Sci. 36 (1996) 420–428.