A minimalist approach to conceptualization of time in quantum theory

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Abstract

Ever since Schrödinger, Time in quantum theory is postulated Newtonian for every reference frame. With mathematical rigor, we show that the concept of the so-called Local Time allows avoiding the postulate. In effect, time appears as neither fundamental nor universal on the quantum-mechanical level while being consistently attributable to every, at least approximately, closed quantum system as well as to every of its (conservative or not) subsystems.

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1 Introduction

Schrödinger's Quantum Mechanics in [1, 2, 3], is timeless when he introduced his fundamental equation as a time-independent equation

$$H\psi = E\psi. \tag{1}$$

Here $E \in \mathbb{R}$ and the Hamiltonian H is of the form

$$H = \frac{\hbar^2}{2m}p^2 + V(x), \quad V(x) = -\frac{e^2}{|x|}, \tag{2}$$

where

$$p = \frac{1}{i}\frac{\partial}{\partial x} = \frac{1}{i}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \tag{3}$$

is the momentum operator conjugate to the position operator $x = (x_1, x_2, x_3)$. With this stationary Schrödinger equation, he could successfully give an explanation of the spectral structure of hydrogen atoms, showing that his formulation of quantum mechanics as the eigenvalue problem of a partial differential operator is valid. Later he proved in [4] that his

formulation is equivalent with Heisenberg's formulation of QM. Without loss of generality, we assume $m = 1$ later on.

In the subsequent part [5] he emphasized the necessity to give a time-dependent expression of the equation in order to treat the nonconservative systems, and gave a time-dependent equation for general Hamiltonians

$$\frac{\hbar}{i}\frac{d\psi}{dt}(t) + H\psi(t) = 0. \tag{4}$$

Schrödinger then applied the equation to some time-dependent perturbations with an emphasis of the advantage of the time-dependent approach. He however gave no justification for the notion of time which is assumed for the equation. That is, "time" is postulated [5] to be unique and universally valid throughout the universe as Newton put it in his *Principia Mathematica*.

Exactly the same physical nature of time is assumed for the standard text-book approach to quantum dynamics that is based on the unitary operator $U(t)$, which defines a dynamical map for quantum systems, $\Psi(t) = U(t)\Psi(t = 0)$. Hence we can detect the following two assumptions (postulates) built in the fundamental equation for quantum systems dynamics. The first assumption is the equation's mathematical form provided by eq.(4), which here we adopt without modification. The second assumption is that quantum dynamics unfolds within the classical Newtonian universal (global) time. However, at least as a logical possibility, removing the second assumption is not excluded and, if successful, might make the quantum foundations even more efficient–the less number of postulates, the better theory.

Avoiding this assumption is not a trivial task, which we undertake in this paper. Rejecting the in-advance-agreed role of "physical time" for the parameter t in the unitary operator $U(t)$ elevates to the following two related problems. First, if not in advance, then certainly *a posteriori* the role of the parameter t as physical time should be rigorously established; non-rigorous procedures typically assume certain additional rules and assumptions, often of the interpretational relevance, that here we are not interested in. Second, without a postulate or an interpretational framework, it is not obvious how to link the time-independent Hamiltonian of closed system with the notion of time. These subtle points are regarded with mathematical rigor in Sections 4 and 5 with the general mathematical basis provided in Section 2. As a result, in Sections 3 and 4 we emphasize a possibility to introduce a notion of time for an arbitrary (including many-particle) *closed* system with the *time-independent* Hamiltonian. We perform *without* resorting to any *ad hoc* procedures or additional assumptions – such as existence of the system's environment, be it classical [6] or not, or time quantization [7], or in-advance-agreed character of physical time. Expectably, such possibility comes at certain price, which in our approach is

that time is neither fundamental nor universal on the quantum-mechanical level, and can be recognized as the so-called (quantum-mechanical) local time [8].

2 N-particle system

In this section we consider a general *conservative* (i.e. *closed*) quantum mechanical system consisting of N particles; we take a unit system such that $\hbar = 1$. For such a system of $N(\geq 2)$ quantum mechanical particles with mass $m_i(>0)$ located at $r_i \in \mathbb{R}^3$ $(i = 1,\ldots,N)$, Hamiltonian (2) becomes

$$\begin{aligned} H = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{r_i} + V(x) = \sum_{i=1}^{N} \frac{1}{2m_i}\left(\frac{1}{i}\frac{\partial}{\partial r_i}\right)^2 + V(x), \\ V(x) = \sum_{1\leq i<j\leq N} V_{ij}(x_{ij}), \end{aligned} \tag{5}$$

where $\Delta = \Delta_{r_i} = \left(\frac{\partial}{\partial r_i}\right)^2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial r_{ij}^2}$ $(r_i = (r_{i1}, r_{i2}, r_{i3}) \in \mathbb{R}^3)$ is Laplacian and $V_{ij}(x_{ij})$ $(x_{ij} = r_i - r_j)$ is a pair potential working between the pair of particles i and j. When we consider the relative motion of N particles, we can separate the motion of the center of mass as follows. The center of mass of this N-particle system is

$$X_C = \frac{m_1 r_1 + \cdots + m_N r_N}{m_1 + \cdots + m_N}. \tag{6}$$

Defining the Jacobi coordinates by

$$\begin{aligned} x_i = (x_{i1}, x_{i2}, x_{i3}) = r_{i+1} - \frac{m_1 r_1 + \cdots + m_i r_i}{m_1 + \cdots + m_i} (\in \mathbb{R}^3), \\ (i = 1,\ldots,N-1) \end{aligned} \tag{7}$$

and corresponding conjugate momentum operators by

$$P_C = \frac{1}{i}\frac{\partial}{\partial X_C}, \quad p_i = \frac{1}{i}\frac{\partial}{\partial x_i} = \frac{1}{i}\left(\frac{\partial}{\partial x_{i1}}, \frac{\partial}{\partial x_{i2}}, \frac{\partial}{\partial x_{i3}}\right), \tag{8}$$

we decompose the Hilbert space $L^2(\mathbb{R}^{3N})$ as a tensor product $L^2(\mathbb{R}^{3N}) = L^2(\mathbb{R}^3) \otimes \mathcal{H}$, $\mathcal{H} = L^2(\mathbb{R}^{3n})$ with $n = N-1$. Accordingly the Hamiltonian H in (5) is decomposed as

follows.

$$
\begin{aligned}
&H = H_C \otimes I + I \otimes \tilde{H},\\
&\tilde{H} = \tilde{H}_0 + V, \quad H_C = \frac{1}{\sum_{j=1}^{N} m_j} P_C^2,\\
&\tilde{H}_0 = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2,\\
&\mu_i^{-1} = m_{i+1}^{-1} + (m_1 + \cdots + m_i)^{-1} \quad (i = 1, \ldots, n).
\end{aligned}
\tag{9}
$$

Here I denotes the identity operator. For real potentials $V_{ij}(x_{ij})$, H in (5) and $\tilde{H}$ in (9) define self-adjoint operators in the Hilbert spaces $L^2(\mathbb{R}^{3N})$ and $\mathcal{H} = L^2(\mathbb{R}^{3n})$, respectively, and the relative motion of the N-particles is described by the Hamiltonian $\tilde{H}$ in $\mathcal{H} = L^2(\mathbb{R}^{3n})$.

By (9), $H_C \otimes I$ is a nonnegative selfadjoint operator in $L^2(\mathbb{R}^3)$ and describes the free motion of the center of mass of the N-particle system whose property is well-known. Our main concern is thus about the relative motion of the N particles. Henceforth we will write

$$H = \tilde{H}, \quad H_0 = \tilde{H}_0, \tag{10}$$

and consider the Hamiltonian in $\mathcal{H} = L^2(\mathbb{R}^{3n})$

$$H = H_0 + V = \sum_{i=1}^{N-1} \frac{1}{2\mu_i} p_i^2 + V(x). \tag{11}$$

We note that H is defined solely through the configuration operators $x = (x_1, \ldots, x_{N-1})$ and conjugate momentum operators $p = (p_1, \ldots, p_{N-1})$. Thus time-independent QM is completely determined through position and momentum operators (x, p), since the corresponding stationary time-independent Schrödinger equation (1) is written as follows.

$$(H - \lambda I)\psi = 0. \tag{12}$$

This equation has non-zero solution $\psi \in \mathcal{H}$ only when λ is an eigenvalue of H: $\lambda \in \sigma_p(H)$. A complex number λ is said to belong to the resolvent set $\rho(H)$, when (12) has only a trivial solution $f = 0$ and the bounded inverse $(H - \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ exists. $R(\lambda) = R_H(\lambda) = (H - \lambda I)^{-1}$ is called the resolvent at $\lambda \in \rho(H)$ of H. We review some concepts on spectrum $\sigma(H)$ of a selfadjoint operator H.

Definition 1. 1) The set of all complex numbers $\lambda \in \mathbb{C} \setminus \rho(H)$ is called the spectrum of H and denoted by $\sigma(H)$. For a selfadjoint operator H it is trivial to see that $\sigma(H) \subset \mathbb{R}$.

2) We denote the resolution of the identity corresponding to a selfadjoint operator H by $E_H(\lambda)$ $(\lambda \in \mathbb{R})$:

$$\begin{aligned} &E_H(\lambda)E_H(\mu) = E_H(\min(\lambda,\mu)), \\ &\text{s-}\lim_{\lambda\to-\infty} E_H(\lambda) = 0, \quad \text{s-}\lim_{\lambda\to\infty} E_H(\lambda) = I, \\ &E_H(\lambda+0) = E_H(\lambda), \\ &f(H) = \int_{-\infty}^{\infty} f(\lambda)dE_H(\lambda) \quad (\forall f \in C(\mathbb{R})), \end{aligned} \tag{13}$$

where $E_H(\lambda+0) = \text{s-}\lim_{\mu\downarrow\lambda} E_H(\mu)$ and $C(\mathbb{R})$ is the set of all complex-valued continuous functions on $\mathbb{R}$. An operator-valued measure $E_H(B)$ $(B \subset \mathbb{R}$: Borel set$)$ is defined by the relation $E_H((a,b]) = E_H(b) - E_H(a)$ for $-\infty < a < b < +\infty$.

3) Set $P(\lambda) = E_H(\lambda) - E_H(\lambda-0)$ $(\lambda \in \mathbb{R})$. We note that $P(\lambda) \neq 0$ iff λ is an eigenvalue of H. When $\lambda \in \sigma_p(H)$, $P(\lambda)\mathcal{H}$ is the eigenspace of H for $\lambda \in \sigma_p(H)$. The pure point spectral subspace (or eigenspace) $\mathcal{H}_p(H)$ for H is defined as the closed linear hull of the set

$$\bigcup_{\lambda\in\mathbb{R}} P(\lambda)\mathcal{H}. \tag{14}$$

Eigenprojection P_H is the orthogonal projection onto $\mathcal{H}_p(H)$.

4) The continuous spectral subspace for H is defined by

$$\begin{aligned} \mathcal{H}_c(H) = \{\psi \mid &E_H(\lambda)\psi \\ &\text{is continuous with respect to } \lambda \in \mathbb{R}\}, \end{aligned} \tag{15}$$

and the absolutely continuous spectral subspace for H by

$$\begin{aligned} \mathcal{H}_{ac}(H) = \{\psi \mid &\text{The measure}(E_H(B)\psi,\psi) = \\ &= \|E_H(B)\|^2 \text{is absolutely continuous with} \\ &\text{respect to Lebesgue measure on } \mathbb{R}\}. \end{aligned} \tag{16}$$

The singular continuous spectral subspace $\mathcal{H}_{sc}(H)$ is defined by $\mathcal{H}_{sc}(H) = \mathcal{H}_c(H) \ominus \mathcal{H}_{ac}(H)$. Then the relation $\mathcal{H} = \mathcal{H}_p(H) \oplus \mathcal{H}_c(H) = \mathcal{H}_p(H) \oplus \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{sc}(H)$ holds.

5) The part H_p, H_c, H_{ac}, H_{sc} of H in $\mathcal{H}_p(H), \mathcal{H}_c(H)$, $\mathcal{H}_{ac}(H), \mathcal{H}_{sc}(H)$ are called spectrally discontinuous, spectrally continuous, spectrally absolutely continuous and

spectrally singular continuous, respectively. The spectra $\sigma(H_p), \sigma(H_c)$, $\sigma(H_{ac})$, $\sigma(H_{sc})$ are called point spectrum, continuous spectrum, absolutely continuous spectrum and singular continuous spectrum of H, and denoted by $\sigma_p(H), \sigma_c(H), \sigma_{ac}(H), \sigma_{sc}(H)$, respectively.

For rather general pair potentials $V_{ij}(x_{ij})$, it is known ([9]) that the singular continuous spectrum $\sigma_{sc}(H)$ is absent: $\mathcal{H}_{sc}(H) = \{0\}$. Therefore we assume henceforth that $\mathcal{H}_c(H) = \mathcal{H}_{ac}(H)$ and $\mathcal{H} = \mathcal{H}_p(H) \oplus \mathcal{H}_{ac}(H)$ hold.

The resolution of the identity $\{E_H(\lambda)\}_{\lambda\in\mathbb{R}}$ gives the spectral property of the selfadjoint operator H, and completely determines H in time-independent manner. In this sense it gives a stationary formulation of the QM system $(H, \mathcal{H})$ with the Hamiltonian H in a Hilbert space $\mathcal{H}$.

3 A missing link in the Schrödinger's approach

In order to illustrate the idea behind our approach to quantum dynamics, let us return to Schrödinger's thoughts nevertheless without historical rigor. Schrödinger [5] starts with wave equation

$$\Delta\psi - \frac{2(E-V)}{E^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{17}$$

As the energy factor E suggests, he has been implicitly assuming the relation (bearing in mind $\hbar = 1$)

$$\psi \sim \mathrm{Re}\,\left(e^{\pm iEt}\right), \tag{18}$$

which he regarded equivalent to

$$\frac{\partial^2\psi}{\partial t^2} = -E^2\psi \tag{19}$$

or (if complex-valued wave function ψ is permitted)

$$\frac{\partial\psi}{\partial t} = \pm iE\psi. \tag{20}$$

From (17) and (19) one has time-independent Schrödinger equation:

$$\left(-\frac{1}{2}\Delta + V - E\right)\psi = 0. \tag{21}$$

Substituting (20) gives time-dependent equation

$$\frac{1}{i}\frac{\partial\psi}{\partial t} \pm \left(-\frac{1}{2}\Delta + V\right)\psi = 0. \tag{22}$$

When ψ satisfies one of the equations (22), the complex conjugate $\overline{\psi}$ satisfies the other, so that one can adopt one of the equations as time-dependent Schrödinger equation:

$$\frac{1}{i}\frac{\partial\psi}{\partial t}+\left(-\frac{1}{2}\Delta+V\right)\psi=0. \tag{23}$$

Even if disregarding some curious points about the assumptions (18)-(20), we note that there is a large discrepancy between the starting equation (17) and the resulting equation (23): (17) is a wave equation and the wave function $\psi(t)$ propagates with a constant velocity $\sqrt{\frac{E^2}{2(E-V)}}$ if we ignore that it might be a complex number. However, eq.(23) is *not* a wave equation and should, in turn, somehow describe also the particle-aspect of quantum systems, in the sense of the standard formalism based on the fundamental position (for brevity denoted x) and momentum (denoted p) observables, which provide the ultimate basis for defining the system's Hamiltonian. Thus time-dependent Schrödinger equation (23) does not describe the wave function propagating with constant velocity. Hence a missing link in the derivation of the fundamental equation (23) for *closed* systems.

As emphasized in Introduction, Time is generally thought to be unique and valid throughout the universe, and when we admit the missing link between equations (17) and (23), one usually regards it a problem of the choice of equation under a given universal time. In this framework of thought, Schrödinger had chosen (23) without giving any justification for the choice.

However if we see the problem closely, we will notice that we can see it as the problem *which notion of time* we should choose. For illustration let us suppose that $V=0$ and $E=2$ for the time being. Then equations (17) and (23) can be written respectively as follows.

$$\frac{1}{i}\frac{\partial\psi}{\partial t}+H^{(1)}\psi=0,\quad H^{(1)}=(-\Delta)^{1/2}, \tag{24}$$

$$\frac{1}{i}\frac{\partial\psi}{\partial t}+H^{(2)}\psi=0,\quad H^{(2)}=-\frac{1}{2}\Delta. \tag{25}$$

Comparing (24) and (25), we see that the rates of change of the state ψ with respect to the same change of time t are different between the two equations. The rate for (24) is

$$H^{(1)}=(-\Delta)^{1/2} \tag{26}$$

and that for (25) is

$$H^{(2)}=-\frac{1}{2}\Delta. \tag{27}$$

However, we can also approach this from the following perspective. We can assume that both equations (24) and (25) are *correct* while describing *different processes* for the

systems that are subject to *different times*, which are generated by the respective Hamiltonians (26) and (27). Hence the new role of the Hamiltonian: instead of solely determining the rate of change of the system's state, the system's Hamiltonian is recognized also to determine the time, which the system is subjected to.

4 Local time and clock, and a justification of the notion of time in quantum mechanics

Generalizing the above argument, we will define the time of a system with Hamiltonian H in accord with [8] as follows. We will call $(H,\mathcal{H})$ a local system when a selfadjoint Hamiltonian H of a closed system is given in a Hilbert space $\mathcal{H}$. Then we can differentiate time among different systems, and the time which is valid only for a *single closed* QM system $(H,\mathcal{H})$ will be called the local time for the system.

To make the situation clear, we define local clock and local time for a quantum mechanical system with Hamiltonian H in (11) as follows. We note that H is a selfadjoint operator defined in a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{3n})$.

Definition 2. The unitary group e^{-itH} is called a *local clock* of the local (closed) system $(H,\mathcal{H})$. The parameter t in the exponent of the local clock e^{-itH} is called the *(quantum mechanical) local time* for the system $(H,\mathcal{H})$.

Essential in Definition 2 is that it does *not in advance* establish the physical meaning of the continuous, real parameter t, which is dubbed "local time". Definition 2 only postulates the unitary dynamical map e^{-itH}, which is generated by the closed system's Hamiltonian with the necessarily appearing a c-number denoted t. Formally it is clear that $\psi(t) = e^{-itH}\psi$ satisfies the time-dependent Schrödinger equation

$$\frac{1}{i}\frac{d\psi}{dt}(t) + H\psi(t) = 0, \quad \psi(0) = \psi, \tag{28}$$

which shows that t introduced in Definition 2 exactly plays the role of time for the system $(H,\mathcal{H})$ as it is assumed by equation (4). However, the physical role of t as physical time is yet to be established.

Another nonstandard element implicit to Definition 2 follows from the fact, that the quantum Universe as we currently perceive it consists of more than one closed ("local") system, each of which *independently* satisfying the conditions of Definition 2. Hence if the parameter t in Definition 2 plays the role of physical time, the Universe consists of plenty of (at least approximately) closed, i.e. local, systems, each of which bearing its own local time generated by their respective local Hamiltonians. To this end, a word of caution is

in order. If we regard the quantum Universe as the only truly closed quantum system, i.e. if we do not allow for at least approximate closed-ness of certain subsystems, we will not be able to describe even a single act of quantum measurement within the unitary quantum theory.

We strongly emphasize that the above introduction of time – that still requires a rigorous procedure of Theorem 1 below – for a system $(H, \mathcal{H})$ has been done with only using the notion of the *time-independent* configuration and momentum operators (x, p), since time t is defined solely through the use of a local system's Hamiltonian H in (11) which is defined by (x, p). In this sense, the notion of time is not, i.e. not necessarily, any fundamental notion of universal importance even if quantum mechanics can be formulated in time-dependent fashion with using time-dependent Schrödinger equation as a basic equation. Therefore a shift in the paradigm of Time [8]: (a) we start from the time-less position and momentum operators, which (b) define the time-independent Hamiltonian H, which generates dynamics of a closed system in Definition 2 and (c) introduces time as an emergent property of the local system with the link (d) one Hamiltonian, one local time for the local (closed) system. In other words: the standard fundamental role of the universal time is abandoned *due to* establishing the fundamental role of the local system's dynamics, i.e. of the local system's clock, Definition 2.

Hence the concept of local time provides a missing link in the original Schrödinger's thoughts by introducing (local) time for a closed quantum system in a consistent way. If the parameter t may be regarded as a closed system's (local) time, then eq.(28) is the differential form of the universal fundamental dynamical law for closed systems that for different Hamiltonians produces different local times, i.e. different dynamics, such as those given by eqs.(24)-(25). Therefore there is no need to choose between the dynamical equations (24) and (25)–they are both correct for their respective local times. Needless to say, eq.(28) straightforwardly leads to derivation of the time-independent equation (1), which now becomes a special case, i.e. non-fundamental physical law. However, bearing in mind that, at its best, equation (28) can serve as a symptom of the physical nature of the parameter t as the physical time for local system, our argument requires the following completion.

We now turn to the nature of local time which tells that the name "time" is appropriate for t. For simplicity we here consider the two-body case $N = 2$ only, whose proof is found in Lemma 5.2 in [10]. For general $N \geq 2$, see Theorem 1 in [8], Theorem 3.2 in [11].

Theorem 1. *Let* $\psi \in \mathcal{H}_c(H)$ *with* $(1+|x|)^2\psi \in \mathcal{H} = L^2(\mathbb{R}^3)$. *Then there is a sequence*

$t_m \to \infty$ $(m \to \infty)$ *such that for any* $\varphi \in C_0^\infty(\mathbb{R})$ *and* $R > 0$

$$\|\chi_{\{x\in\mathbb{R}^3||x|<R\}} e^{-it_m H}\psi\| \to 0, \tag{29}$$

$$\|(\varphi(H) - \varphi(H_0))e^{-it_m H}\psi\| \to 0, \tag{30}$$

$$\left\| \left(\frac{x}{t_m} - \frac{p}{\mu} \right) e^{-it_m H}\psi \right\| \to 0 \tag{31}$$

as $m \to \infty$*, where* $p = -i\partial/\partial x$*,* μ *is reduced mass, and* χ_B *denotes the characteristic function of a set B. The similar asymptotic relations hold for some sequence* $t_m \to -\infty$ $(m \to -\infty)$.

We note that x denotes the distance operator at "time" t from the origin around which the quantum particle is assumed to have started at the initial "time" $t = 0$. Thus the theorem tells that for a scattering state ψ belonging to the continuous spectral subspace $\mathcal{H}_c(H)$ for H, the local clock e^{-itH} works such that the "mechanical" velocity x/t_m becomes close to quantum mechanical velocity $p/\mu = -\mu^{-1}i\partial/\partial x$ as $m \to \infty$ on the state $e^{-it_m H}\psi$.

$$\frac{x}{t_m} \sim \frac{p}{\mu} \quad (t_m \to \infty). \tag{32}$$

This tells that the quantum mechanical wave function $e^{-it_m H}\psi$ travels most densely around a trajectory of a classical counterpart for which eq.(32) would be equivalent with the classical time expressed in the well-known form, $t = \mu x/p$, which, in turn, is sometimes used as a basis of time quantization [7].

Hence the following answer to the first problem indicated in Section 1: the c-number t for a local clock e^{-itH} assumes the role of "time" from classical mechanics.

5 Fourier-Laplace transform of a local clock

Theorem 1 indicates that a closed system's Hamiltonian generates dynamics, which, in turn, bears the system's local time. In this section, we show that the inverse also holds, that is, we show that local time t established by Theorem 1 determines the time-independent Hamiltonian of a closed system, that answers the second problem indicated in Section 1.

First assume that $\mathcal{H}_c(H) = \{0\}$. Then one has $\mathcal{H} = \mathcal{H}_p(H)$ and thus the space $\mathcal{H}$ is spanned by just the eigenfunctions ψ for H. Hence the spectral property of H is completely determined by timeless Schrödinger equation (1). Appearance of time also for this case regards the generic state $\psi = \sum_{j=1}^K a_j\psi_j$, with $H\psi_j = E_j\psi_j$, $\psi_j \neq 0$, and thus $|e^{-itH}\psi(x)|^2 = \sum_{j,k=1}^K e^{-it(E_j-E_k)}a_j\overline{a}_k\psi_j(x)\overline{\psi}_k(x) \neq$ constant in general. This describes the

operation of a local clock on $\psi \in \mathcal{H}_p(H)$. Therefore, what Schrödinger did in [1, 2, 3] is to identify eigenvalues E and corresponding eigenfunctions ψ of H. Thus his work in those papers is an analysis of pure point spectrum $\sigma_p(H)$ of H and the eigenspace $P(\lambda)\mathcal{H}$ for H with $\lambda \in \sigma_p(H)$. This result clarified the structure of H on the eigenspace $\mathcal{H}_p(H) \subset \mathcal{H}$ of H. In other words, the time-dependent analysis of e^{-itH} on $\mathcal{H}_p(H)$ is reduced to the time-independent analysis of eigenvalues and the corresponding eigenfunctions of H.

The inverse to this is as follows. It is known that the following mean ergodic identity holds for each $\lambda \in \mathbb{R}$ (see Ch. 10 in [12]).

$$P(\lambda) = \text{s-}\lim_{t_2-t_1\to\infty}(t_2-t_1)^{-1}\int_{t_1}^{t_2} e^{-it\lambda}e^{itH}dt. \tag{33}$$

Thus analysis of the time-independent Hamiltonian is reduced to the analysis of the solution e^{-itH} of time-dependent equation (4).

The analysis of other spectrum $\sigma_{ac}(H) = \sigma_c(H)$ of H is reduced to the analysis of the absolutely continuous part H_{ac} of H, i.e. to the analysis of H restricted to the absolutely continuous subspace $\mathcal{H}_{ac}(H) = \mathcal{H}_c(H)$. As the measure $(E_H(B)\psi,\psi) = \|E_H(B)\psi\|^2$ is absolutely continuous for $\psi \in \mathcal{H}_{ac}(H)$, there exists an integrable differentiation $\frac{d}{d\lambda}(E(\lambda)\psi,\psi)$ for all $\lambda \in \mathbb{R}$ such that for a Borel set B of $\mathbb{R}$ the following relation holds.

$$(E_H(B)\psi,\psi) = \int_B \frac{d}{d\lambda}(E(\lambda)\psi,\psi)d\lambda \quad (\psi \in \mathcal{H}_{ac}(H)). \tag{34}$$

Let $\tilde{\mathcal{T}}$ be the closed set of all eigenvalues of H and its subsystem Hamiltonians. Then it is known (see Theorem 8.1 in [9]) that for $\psi \in L^2_\delta(\mathbb{R}^{3n})$ $(1 \geq \delta > 1/2)$ and $\lambda \in \mathbb{R}\setminus\tilde{\mathcal{T}}$, the boundary value $R(\lambda \pm i0)$ as $\varepsilon \downarrow 0$ of the resolvent $R(\lambda \pm i\varepsilon)$ exists as a bounded operator from a subspace $L^2_\delta(\mathbb{R}^{3n})(\subset L^2(\mathbb{R}^{3n}))$ into its dual space $L^2_{-\delta}(\mathbb{R}^{3n})$, and satisfies for $\psi \in L^2_\delta(\mathbb{R}^{3n})$

$$\frac{dE}{d\lambda}(\lambda)\psi = \frac{1}{2\pi i}(R(\lambda+i0)-R(\lambda-i0))\psi \in L^2_{-\delta}(\mathbb{R}^{3n}). \tag{35}$$

In general, by (13) local clock $e^{-itH}\psi$ for $\psi \in \mathcal{H}_{ac}(H)$ is given by a Fourier transform of $dE(\lambda)$ so that we have

$$\begin{aligned} e^{-itH}\psi &= \int_{\mathbb{R}} e^{-it\lambda}dE(\lambda)\psi \\ &= \int_{\mathbb{R}} e^{-it\lambda}\frac{dE}{d\lambda}(\lambda)\psi d\lambda \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda}(R(\lambda+i0)-R(\lambda-i0))\psi d\lambda. \end{aligned} \tag{36}$$

This shows that the analysis of time evolution or local clock $e^{-itH}\psi$ of the system with Hamiltonian H for $\psi \in \mathcal{H}_{ac}(H)$ can be reduced to the analysis of the boundary values of the resolvent $R(\lambda \pm i\varepsilon)\psi$ as $\varepsilon \downarrow 0$. Hence time-dependent analysis of quantum mechanics on $\mathcal{H}_{ac}(H)$ can be derived from time-independent analysis of QM.

Conversely, writing $R(z) = (H-z)^{-1} = (H_0+V-z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\begin{aligned} R(z)\psi = (H-z)^{-1}\psi = \pm i \int_0^{\pm\infty} e^{itz} e^{-itH}\psi dt \\ (\pm \mathrm{Im}\, z > 0, \psi \in \mathcal{H}). \end{aligned} \tag{37}$$

This shows that the analysis of the boundary values of the resolvent $R(z)$ is reduced to the analysis of the convergence of Fourier-Laplace transform of the local clock e^{-itH} when $|\mathrm{Im}\, z| \to 0$. In this sense, the analysis of spectral property of the time-independent Hamiltonian H can be reduced to the analysis of the solution of the time-dependent Schrödinger equation.

These show that the time-dependent analysis and time-independent analysis of QM are equivalent for $\psi \in \mathcal{H}_{ac}(H)$. Together with the result we have shown for $\psi \in \mathcal{H}_p(H)$, these provide the desired argument, which now can be stated as

Theorem 2. *Time-dependent analysis and stationary analysis of Quantum Mechanics are mutually equivalent for closed systems.*

6 Quantum field theory

We consider in this section how the local time works in the case of Quantum Field Theory (QFT). In QFT that ignores the spin of the system, Hamiltonian of a system is given as follows. Let $q(x), p(x)$ be maps from $\mathbb{R}^3$ into a space of selfadjoint operators in a Hilbert space such that the following canonical commutation relations hold for all $x, x' \in \mathbb{R}^3$.

$$\begin{aligned} &[q(x), p(x')] = i\delta(x-x'), \\ &[q(x), q(x')] = [p(x), p(x')] = 0. \end{aligned} \tag{38}$$

Then the Hamiltonian H is defined by

$$H = \frac{1}{2}\int (p(x)^2 + c^2 \nabla q(x)^2 + c^4 \mu^2 q(x)^2) dx. \tag{39}$$

We assume that H defines a selfadjoint operator in a suitable Hilbert space. Then we can define local clock and time of the system by the evolution e^{-itH} as in Definition 2. Using

the local time of the system, we define

$$\begin{cases} q(x,t) = e^{itH} q(x) e^{-itH}, \\ p(x,t) = e^{itH} p(x) e^{-itH}. \end{cases} \tag{40}$$

Let

$$a^{(j)} = (0,\dots,0,\overset{\overset{j}{\smile}}{a},0,\dots,0)$$

be a vector with j-th component being a and others zero. Then recalling that

$$\begin{aligned} \nabla q(x) &= \left(\frac{\partial q}{\partial x_j}\right)_{j=1}^{3}, \\ \frac{\partial q}{\partial x_j}(x) &= \lim_{a\to 0} \frac{q(x+a^{(j)}) - q(x)}{a}, \end{aligned} \tag{41}$$

we have

$$\begin{aligned} \left[\frac{\partial q}{\partial x_j}(x), p(x')\right] &= i\frac{\partial \delta}{\partial x_j}(x-x'), \\ [\nabla q(x), p(x')] &= i\nabla\delta(x-x'). \end{aligned} \tag{42}$$

From this and (39) follows:

Theorem 3. *For (q,p) defined above, we have*

$$\begin{cases} \dfrac{\partial q}{\partial t}(x,t) = p(x,t), \\ \dfrac{\partial p}{\partial t}(x,t) = (c^2\Delta q - c^4\mu^2 q)(x,t). \end{cases} \tag{43}$$

Therefore we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + c^2\mu^2\right) q(x,t) = 0. \tag{44}$$

This holds if the following holds

$$\frac{1}{i}\frac{\partial q}{\partial t}(x,t) + c\sqrt{-\Delta + c^2\mu^2}\, q(x,t) = 0. \tag{45}$$

If we take c equal to the speed of light, the equation (44) becomes covariant with respect to the Lorentz transformations, and thus this equation successfully describes the free relativistic field. The obtained equation (44) is the Klein-Gordon equation and shows that the field propagates as a wave. Furthermore (45) shows that to adopt Hamiltonian H in (39) is equivalent to adopting the Hamiltonian

$$H_{(r)} = c\sqrt{-\Delta + c^2\mu^2} \tag{46}$$

and the local clock $e^{-itH_{(r)}}$ for the same QF system. Thus the corresponding Schrödinger equation

$$\frac{1}{i}\frac{dq}{dt}(t) + H_{(r)}q(t) = 0. \tag{47}$$

is a fundamental equation for free relativistic quantum field theory.

We recall that all this is formally given through the use of Fock space $F = \bigoplus_{k=0}^{\infty} \mathcal{H}^n$ ($\mathcal{H}^n = \overbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}^{n \text{ factors}}$) and a selfadjoint operator in F like

$$H = \sum_{k=0}^{\infty} \omega_k a_k^\dagger a_k, \tag{48}$$

where $\omega_k = c\sqrt{k^2 + c^2m^2}$, and $a_k^\dagger$ and a_k are creation and annihilation operators, respectively.

7 Discussion

Modern open quantum systems theory [13, 14] offers a unique physical basis for the explicit appearance of time for non-conservative systems. Time dependence of the system's Hamiltonian may be due to the environmental influence, e.g. [6]. That is, explicit time dependence in a quantum system's Hamiltonian may be a symptom of the system's interaction with another system that is often called environment. Hence whenever we start with arguments regarding open systems, we may ultimately end up with a closed system [6, 13, 14] that is described by the fundamental dynamical law (4) and hence with the conclusion that all subsystems (conservative or not) of a closed system share the same physical time.

As we emphasized in Introduction, the standard global and universal time common for all subsystems (degrees of freedom) of the quantum Universe appears as an assumption additional to the fundamental postulates of quantum theory. Bearing in mind Definition 2,

this assumption is, in our opinion, a huge step requiring justification or otherwise becomes Procrustean.

On the other hand, the concept of emergent local time, Definition 2, neither relies nor it requires any assumptions additional to the postulate of the fundamental unitary dynamics in quantum theory. Local Time paradigm [8] establishes physical time for a single closed system without any intrinsic inconsistencies, which are otherwise found for some concurrent approaches to the concept of local time (or 'multi time') in the non-relativistic context [15].

Some details and ramifications regarding the concept of local time can be found in Refs. [8, 16, 17] while certain corollaries of Local Time paradigm can be found in [17, 18]. Interpretational consequences and links with the existing approaches to time in quantum theory will be presented elsewhere.

8 Conclusion

The concept of Local Time [8] is a minimalist alternative to the standard concept of universal time in the unitary quantum theory. The unitary dynamics bears local time as an internal characteristic that is neither fundamental nor universal on the quantum-mechanical level while being consistently attributable to every, at least approximately, closed quantum system.

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