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Research Article

Fixed Point Results Satisfying Rational Type Contraction in G-Metric Spaces

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A unique fixed point theorem for three self-maps under rational type contractive condition is established. In addition, a unique fixed point result for six continuous self-mappings through rational type expression is also discussed.

1. Introduction

Fixed point theory is one of the core subjects of nonlinear analysis. This theory is not constrained to mathematics; it is also applicable to other disciplines. It is closely linked with social and medical science, military applications, graph theory [1], game theory, economics [2], statistics, and medicine. This theory is divided into three categories: topological fixed point theory, metric fixed point theory, and discrete fixed point theory.

In metric fixed point theory, the first result proved by Banach [3] is known as Banach contraction principle. Many researchers extended this principle for the study of fixed points and common fixed points using different types of contraction such as weak contraction [4, 5], integral type contraction [6], rational type contraction [7], and T-Hardy Rogers type contraction [8]. For more details, see [9–11] and so forth.

Dass and Gupta [12] gave the extension of Banach's contraction mapping principle by using a contractive condition of rational type. Jaggi [7] proved some unique fixed point results through contractive condition of rational type in metric spaces. Harjani et al. [13] studied the results of Jaggi in the setting of partially ordered metric spaces. Using generalized weak contractions Luong and Thuan [14] generalized the results of [13] through rational type expressions in the context of partially ordered metric spaces. Chandok and Karapinar [15] generalized the results of Harjani and established common fixed point results for weak contractive conditions satisfying rational type expressions in partially ordered metric spaces. Mustafa et al. [16] discussed fixed point results by almost generalized contraction via rational type expression which generalizes, extends, and unifies the results of Jaggi [7], Harjani et al. [13], and Luong and Thuan [14], respectively. Fixed point theorems for contractive type conditions satisfying rational inequalities in metric spaces have been developed in a number of works; see [17–20] and so forth.

Mustafa and Sims [21] generalized the notion of metric space as an appropriate notion of generalized metric space called G-metric space. They have investigated convergence in G-metric spaces, introduced completeness of G-metric spaces, and proved a Banach contraction mapping theorem and some other fixed point theorems involving contractive type mappings in G-metric spaces using different contractive conditions. Later, various authors have proved some common fixed point theorems in these spaces (see [8, 22–24]).

Sanodia et al. [25] used rational type contraction and investigated a unique fixed point theorem for single mapping in G-metric spaces. Gandhi and Bajpai [26] generalized the result of Sanodia et al. and proved unique common fixed point results for three mappings in G-metric space satisfying rational type contractive condition. Recently, Shrivastava et al. [27] established some unique fixed point theorem for some new rational type contraction.

The aim of this paper is to establish two common fixed point theorems satisfying rational type contraction. In the first result, we discuss the existence and uniqueness of common fixed point for three self-maps in the context of G-metric space, while in the second one we studied the uniqueness of common fixed point for six continuous self-mappings in the setting of G-metric through rational type expression.

2. Preliminaries

We recall some definitions that will be used in our discussion.

Definition 1 (see [21]). Let X be a nonempty set and let $G : X\times X\times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(1) $G(x, y, z) = 0$ implies that $x = y = z$ for all $x, y, z \in X$.

(2) $G(x, x, y) \le G(x, y, z)$ for all $x, y, z \in X$.

(3) $G(x, y, z) = G(x, z, y) = G(y, z, x) \cdots$ for all $x, y, z \in X$.

(4) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then, it is called G-metric and the pair (X, G) is a G-metric space.

Proposition 2 (see [21]). *Let (X, G) be a G-metric space. The following are equivalent:*

(1) *(x_n) is G-convergent to x.*

(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*

(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Definition 3 (see [22, 28]). A pair of self-mappings f, g in a G-metric space is said to be weakly commuting if

$$G(fgx, gfx, gfx) \le G(fx, gx, gx), \quad \forall x \in X. \tag{1}$$

Sanodia et al. [25] proved the following fixed point theorem in the setting of G-metric space.

Theorem 4. *Let (X, G) be a G-complete G-metric space and let $f : X \to X$ be a self-map satisfying the condition*

$$\begin{aligned} G(fx, fy, fz) &\le A \\ &\cdot \frac{\max\{G^2(x, fx, fy), G^2(y, fy, fz), G^2(z, fz, fx)\}}{G(x, y, z)} \end{aligned} \tag{2}$$

for all $x, y, z \in X$ with $0 \le A < 1$. Then, f has a unique common fixed point in X.

Theorem 5. *Let (X, G) be a G-complete G-metric space and let $S, T : X \to X$ be two self-maps such that $S(X) \subset T(X)$ satisfying the following condition:*

$$\begin{aligned} G(Tx, Ty, Tz) &\le A \\ &\cdot \frac{\max\{G^2(Sx, Tx, Ty), G^2(Sy, Ty, Tz), G^2(Sz, Tz, Tx)\}}{G(Sx, Sy, Sz)} \end{aligned} \tag{3}$$

for all $x, y, z \in X$ with $0 \le A < 1$. Then, S and T have a unique common fixed point in X.

Gandhi and Bajpai [26] proved unique common fixed point results satisfying the following rational type contractive condition.

Theorem 6. *Let (X, G) be a G-complete G-metric space and let $f, g, h : X \to X$ be three self-mappings satisfying the condition*

$$\begin{aligned} G(fx, gy, hz) &\le A \\ &\cdot \frac{\max\{G^2(x, fx, gy), G^2(y, gy, hz), G^2(z, hz, fx)\}}{G(x, y, z)} \end{aligned} \tag{4}$$

for all $x, y, z \in X$ with $0 \le A < 1$. Then, f, g, and h have a unique common fixed point in X.

Currently, Shrivastava et al. [27] studied the following result.

Theorem 7. *Let (X, G) be a G-complete G-metric space and let $f : X \to X$ be a self-map satisfying the condition*

$$\begin{aligned} G(fx, fy, fz) &\le A \cdot \frac{G(x, fy, fy) + G(x, fz, fz)}{2} + B \\ &\cdot (G(x, fy, fy)\, G(x, fy, fy) + G(x, fz, fz) \\ &+ G(y, fx, fx) + G(z, fx, fx)) \\ &\cdot (2(G(x, fy, fy) + G(y, fx, fx)))^{-1} \end{aligned} \tag{5}$$

for all $x, y, z \in X$ with $0 \le A + B < 1/2$. Then, f has a unique common fixed point in X and f is G-continuous at u.

3. Main Results

Our first new result is the following.

Theorem 8. *Let (X, G) be a G-complete G-metric space and let $S, T, R : X \to X$ be three self-mappings satisfying the following condition:*

$$\begin{aligned} G(Sx, Ty, Rz) &\le A \cdot \big(G(x, Sx, Ty)\, G(y, Ty, Rz) \\ &+ [G(x, y, z)]^2 + G(x, Sx, Ty)\, G(x, y, z)\big) \\ &\cdot (G(x, Sx, Ty) + G(x, y, z) + G(y, Ty, Rz))^{-1} \\ &+ B \cdot \big(G(y, Ty, Rz)\,[1 + G(x, Sx, Ty)] \\ &\cdot (1 + G(x, y, z))^{-1}\big) + C \cdot G(x, y, z) \end{aligned} \tag{6}$$

for all $x, y, z \in X$ *with* $x \neq y \neq z \neq x$, $A, B, C \geq 0$ *with* $0 \leq A+B+C < 1$, $G(x, Sx, Ty)+G(x, y, z)+G(x, Ty, Rz) \neq 0$. *Then,* S, T, *and* R *have a common fixed point. Further, if* $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) = 0$ *implies* $G(Sx, Ty, Rz) = 0$, *then* S, T, *and* R *have a unique common fixed point in* X.

Proof. Let x_0 be arbitrary in X; we define a sequence x_n by the following rules:

$$\begin{aligned} x_{3n+1} &= Sx_{3n}, \\ x_{3n+2} &= Tx_{3n+1}, \\ x_{3n+3} &= Rx_{3n+2}, \\ &\forall n \in \mathbb{X}. \end{aligned} \tag{7}$$

Now, we have to show that x_n is a G-Cauchy sequence in X. Consider $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) \neq 0$; from (6), we have

$$\begin{aligned} &G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}) \leq A \\ &\cdot \Big[G(x_{3n}, Sx_{3n}, Tx_{3n+1}) G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) \\ &+ [G(x_{3n}, x_{3n+1}, x_{3n+2})]^2 + G(x_{3n}, Sx_{3n}, Tx_{3n+1}) \\ &\cdot G(x_{3n}, x_{3n+1}, x_{3n+2})\Big] (G(x_{3n}, Sx_{3n}, Tx_{3n+1}) \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &+ G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}))^{-1} + B \\ &\cdot \Big(G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) [1 \\ &+ G(x_{3n}, Sx_{3n}, Tx_{3n+1})] (1 \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2}))^{-1}\Big) + C \cdot G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &= A \cdot \Big[G(x_{3n}, x_{3n+1}, x_{3n+2}) G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &+ [G(x_{3n}, x_{3n+1}, x_{3n+2})]^2 + G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &\cdot G(x_{3n}, x_{3n+1}, x_{3n+2})\Big] (G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}))^{-1} \\ &+ B \cdot \Big(G(x_{3n+1}, x_{3n+2}, x_{3n+3}) [1 \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2})] (1 \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2}))^{-1}\Big) + C \cdot G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &= A \cdot [G(x_{3n}, x_{3n+1}, x_{3n+2}) (G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \\ &+ G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}))] \\ &\cdot (G(x_{3n}, x_{3n+1}, x_{3n+2}) + G(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &+ G(x_{3n+1}, x_{3n+2}, x_{3n+3}))^{-1} + B \\ &\cdot \Big(G(x_{3n+1}, x_{3n+2}, x_{3n+3}) [1 + G(x_{3n}, x_{3n+1}, x_{3n+2})] \\ &\cdot (1 + G(x_{3n}, x_{3n+1}, x_{3n+2}))^{-1}\Big) + C \cdot G(x_{3n}, x_{3n+1}, \\ &x_{3n+2}) = A \cdot G(x_{3n}, x_{3n+1}, x_{3n+2}) + B \cdot G(x_{3n+1}, \\ &x_{3n+2}, x_{3n+3}) + C \cdot G(x_{3n}, x_{3n+1}, x_{3n+2}) = (A + C) \\ &\cdot G(x_{3n}, x_{3n+1}, x_{3n+2}) + B \cdot G(x_{3n+1}, x_{3n+2}, x_{3n+3}), \end{aligned} \tag{8}$$

which implies that

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq h \cdot G(x_{3n}, x_{3n+1}, x_{3n+2}), \tag{9}$$

where $h = (A + C)/(1 - B)$.

Similarly,

$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) \leq h \cdot G(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{10}$$

Therefore, for all n, we have

$$\begin{aligned} G(x_{n+1}, x_{n+2}, x_{n+3}) &\leq h \cdot G(x_n, x_{n+1}, x_{n+2}) \leq \cdots \\ &\leq h^{n+1} \cdot G(x_0, x_1, x_2). \end{aligned} \tag{11}$$

Now, for all l, m, n, with $l > m > n$, using rectangular inequality, the second axiom of the G-metric, and (11), we have

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\ &\quad + G(x_{l-2}, x_{l-1}, x_l) \\ &\leq G(x_n, x_{n+1}, x_{n+2}) \\ &\quad + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots \\ &\quad + G(x_{l-2}, x_{l-1}, x_l) \\ &\leq h^n + h^{n+1} + \cdots + h^{l-2} \cdot G(x_0, x_1, x_2) \\ &= \frac{h^n}{1-h} \cdot G(x_0, x_1, x_2), \end{aligned} \tag{12}$$

where $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

This shows that x_n is a G-Cauchy sequence. But (X, G) is G-complete G-metric space so there exists w in X such that $x_n \to w$ as n tends to infinity.

Now, we assume that $sw \neq w$. Using condition (6), we have

$$\begin{aligned} &G(Sw, x_{3n+2}, x_{3n+3}) = G(Sw, Tx_{3n+1}, Rx_{3n+2}) \leq A \\ &\cdot \Big[G(w, Sw, Tx_{3n+1}) G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) \\ &+ [G(w, x_{3n+1}, x_{3n+2})]^2 + G(w, Sw, Tx_{3n+1}) \\ &\cdot G(w, x_{3n+1}, x_{3n+2})\Big] (G(w, Sw, Tx_{3n+1}) \\ &+ G(w, x_{3n+1}, x_{3n+2}) \\ &+ G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}))^{-1} + B \end{aligned}$$

$$\cdot \Big(G(x_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) \cdot [1 + G(w, Sw, Tx_{3n+1})] \cdot (1 + G(w, x_{3n+1}, x_{3n+2}))^{-1}\Big) + C \cdot G(w, x_{3n+1}, x_{3n+2}) = A \cdot \big[G(w, Sw, x_{3n+2}) \cdot G(x_{3n+1}, x_{3n+2}, x_{3n+3}) + [G(w, x_{3n+1}, x_{3n+2})]^2 + G(w, Sw, x_{3n+2}) G(w, x_{3n+1}, x_{3n+2})\big] \cdot (G(w, sw, x_{3n+2}) + G(w, x_{3n+1}, x_{3n+2}) + G(x_{3n+1}, x_{3n+2}, x_{3n+3}))^{-1} + B \cdot \Big(G(x_{3n+1}, x_{3n+2}, x_{3n+3}) [1 + G(w, Sw, x_{3n+2})] \cdot (1 + G(w, x_{3n+1}, x_{3n+2}))^{-1}\Big) + C \cdot G(w, x_{3n+1}, x_{3n+2}). \tag{13}$$

As x_n is G-Cauchy sequence and converges to w, therefore, by taking limit $n \to \infty$, we get $G(Sw, w, w) \le 0$ which is held only if $G(Sw, w, w) = 0$ implies that $Sw = w$. Similarly, it can be shown that $Tw = w$ and $Rw = w$. Hence, w is a common fixed point of S, T and R.

Uniqueness. Suppose that S, T, and R have two common fixed points z and w such that $z \neq w$. Since condition $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) = 0$ implies $G(Sx, Ty, Rz) = 0$, we have that $G(z, Sz, Tw) + G(z, w, w) + G(z, Tw, Rw) = 0$ implies $G(Sz, Tw, Rw) = 0$. Therefore, one can get the following:

$$G(Sz, Tw, Rw) = G(z, w, w) = 0 \quad \text{implies that } z = w, \tag{14}$$

which is a contradiction. Therefore, the common fixed point is unique. □

Corollary 9. *Let (X, G) be a G-complete G-metric space and let $S, T, R : X \to X$ be three self-mappings satisfying the condition*

$$G(Sx, Ty, Rz) \le A \cdot \big[G(x, Sx, Ty) G(x, Ty, Rz) + [G(x, y, z)]^2 + G(x, Sx, Ty) G(x, y, z)\big] \cdot (G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz))^{-1} \tag{15}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$ $A \ge 0$ with $0 \le A < 1$, $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) \neq 0$. Then, S, T, and R have a common fixed point. Further, if $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) = 0$ implies $G(Sx, Ty, Rz) = 0$, then S, T, *and R have a unique common fixed point in X.*

Proof. The proof follows by taking $B = C = 0$ in Theorem 8. □

Corollary 10. *Let (X, G) be a G-complete G-metric space and let $S, T, R : X \to X$ be three self-mappings satisfying the condition*

$$G(Sx, Ty, Rz) \le B \cdot \frac{G(y, Ty, Rz)[1 + G(x, Sx, Ty)]}{1 + G(x, y, z)} + C \cdot G(x, y, z) \tag{16}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$ $B, C \ge 0$ with $0 \le B + C < 1$, $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) \neq 0$. Then S, T, and R have a common fixed point. Further, if $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Rz) = 0$ implies $G(Sx, Ty, Rz) = 0$, then S, T, and R have a unique common fixed point in X.

Proof. The proof follows by taking $A = 0$ in Theorem 8. □

Corollary 11. *Let (X, G) be a G-complete G-metric space and let $S, T : X \to X$ be two self-mappings satisfying the condition*

$$G(Sx, Ty, Tz) \le A \cdot \big[G(x, Sx, Ty) G(x, Ty, Tz) + [G(x, y, z)]^2 + G(x, Sx, Ty) G(x, y, z)\big] \cdot (G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Tz))^{-1} + B \cdot \Big(G(y, Ty, Tz)[1 + G(x, Sx, Ty)] \cdot (1 + G(x, y, z))^{-1}\Big) + C \cdot G(x, y, z) \tag{17}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$ $A, B, C \ge 0$ with $0 \le A + B + C < 1$, $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Tz) \neq 0$. Then, S and T have a common fixed point. Further, if $G(x, Sx, Ty) + G(x, y, z) + G(x, Ty, Tz) = 0$ implies $G(Sx, Ty, Tz) = 0$, then S and T have a unique common fixed point in X.

Proof. The proof follows by taking $R = T$ in Theorem 8. □

By setting $R = T = S$ in Theorem 8, we have the following corollary.

Corollary 12. *Let (X, G) be a G-complete G-metric space and let $T : X \to X$ be a self-mapping satisfying the condition*

$$G(Tx, Ty, Tz) \le A \cdot \big[G(x, Tx, Ty) G(x, Ty, Tz) + [G(x, y, z)]^2 + G(x, Tx, Ty) G(x, y, z)\big] \cdot (G(x, Tx, Ty) + G(x, y, z) + G(x, Ty, Tz))^{-1} + B \cdot \Big(G(y, Ty, Tz)[1 + G(x, Tx, Ty)] \cdot (1 + G(x, y, z))^{-1}\Big) + C \cdot G(x, y, z) \tag{18}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$ $A, B, C \ge 0$ with $0 \le A + B + C < 1$, $G(x, Tx, Ty) + G(x, y, z) + G(x, Ty, Tz) \neq 0$. Then, T has a unique fixed point. Further, if $G(x, Tx, Ty) + G(x, y, z) + G(x, Ty, Tz) = 0$ implies $G(Tx, Ty, Tz) = 0$, then T has a unique common fixed point in X.

The second main result in this section is the following.

Theorem 13. *Let (X, G) be a G-complete G-metric space. Let $R, S, T, I, J, Q : X \to X$ be six continuous self-maps and let $\{S, I\}$, $\{T, J\}$, and $\{R, Q\}$ be weakly commuting pairs of self-mapping such that $T(X) \subset I(X)$, $S(X) \subset J(X)$, and $R(X) \subset Q(X)$, satisfying the condition*

$$G(Rx, Sy, Tz) \le A \cdot \left[G(Qx, Sx, Iz) G(Rx, Sx, Ix) + [G(Qx, Jy, Iz)]^2 + G(Rx, Sx, Ix) G(Qx, Jy, Iz)\right] (G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix))^{-1} + B \cdot G(Qx, Jy, Iz) \tag{19}$$

for all $x, y, z \in X$ with $x \ne y \ne z \ne x$ $A, B \ge 0$ with $0 \le A + B < 1$, $G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix) \ne 0$. Then R, S, T, I, J, Q have a common fixed point. Further, if $G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix) = 0$ implies $G(Sx, Ty, Rz) + G(Qx, Jy, Iz) = 0$, then R, S, T, I, J, Q have a unique common fixed point in X.

Proof. Take x_0 as arbitrary point of X. Since $R(X) \subset Q(X)$, we can find a point x_1 in X such that $Rx_0 = Qx_1$. For $S(X) \subset J(X)$, we can find a point x_2 in X such that $Rx_1 = Qx_2$ and for $T(X) \subset I(X)$ we can find a point x_3 in X such that $Tx_2 = Ix_3$. Generally, for a point x_{3n}, choose x_{3n+1} such that $Rx_{3n} = Qx_{3n+1}$; for a point x_{3n+1}, choose x_{3n+2} such that $Sx_{3n+1} = Jx_{3n+2}$; and for a point x_{3n+2}, choose x_{3n+3} such that $Tx_{3n+2} = Ix_{3n+3}$ for $n = 0, 1, 2, 3, \ldots$.

Suppose $G_{3n} = G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) \ne 0$ and $G_{3n+1} = G(Rx_{3n+1}, Sx_{3n+2}, Tx_{3n+3}) \ne 0$. Then, from condition (19), we have

$$\begin{aligned} G_{3n+1} &= G(Rx_{3n+1}, Sx_{3n+2}, Tx_{3n+3}) \le A \\ &\cdot \left[G(Qx_{3n+1}, Sx_{3n+1}, Ix_{3n+3}) \cdot G(Rx_{3n+1}, Sx_{3n+1}, Ix_{3n+1}) + [G(Qx_{3n+1}, Jx_{3n+2}, Ix_{3n+3})]^2 + G(Rx_{3n+1}, Sx_{3n+1}, Ix_{3n+1}) \cdot G(Qx_{3n+1}, Jx_{3n+2}, Ix_{3n+3})\right] \\ &\cdot [G(Rx_{3n+1}, Sx_{3n+1}, Ix_{3n+1}) + G(Qx_{3n+1}, Jx_{3n+2}, Ix_{3n+3}) + G(Rx_{3n+1}, Sx_{3n+1}, Ix_{3n+1})]^{-1} + B \cdot G(Qx_{3n+1}, Jx_{3n+2}, Ix_{3n+3}) = A \\ &\cdot \left[G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) \cdot G(Rx_{3n+1}, Sx_{3n+1}, Tx_{3n}) + [G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2})]^2 + G(Rx_{3n+1}, Sx_{3n+1}, Tx_{3n}) \cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2})\right] \\ &\cdot [G(Rx_{3n+1}, Sx_{3n+1}, Tx_{3n}) + G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) + G(Rx_{3n+1}, Sx_{3n+1}, Tx_{3n})]^{-1} + B \\ &\cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) = A \cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) + B \cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) = (A + B) \\ &\cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}). \end{aligned} \tag{20}$$

Hence,

$$\begin{aligned} G(Rx_{3n+1}, Sx_{3n+2}, Tx_{3n+3}) &\le (A + B) \cdot G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}), \\ G_{3n+1} &\le h \cdot G_{3n}, \end{aligned} \tag{21}$$

where $h = A + B$. Continuing this procedure, in the end we get

$$G_{3n+1} \le h \cdot G_{3n} \le h^2 \cdot G_{3n-1} \le h^3 \cdot G_{3n-2} \le h^4 \cdot G_{3n-3} \le \cdots \le h^{3n+1} \cdot G_0. \tag{22}$$

Clearly, $G_{3n+1} \to 0$ as $n \to \infty$. So, $G(Rx_{3n}, Sx_{3n+1}, Tx_{3n+2}) \to 0$; we get the following sequence:

$$\{Rx_0, Sx_1, Tx_2, Rx_3, Sx_4, Tx_5, Rx_6, Sx_7, Tx_8, \ldots, Rx_{3n+1}, Sx_{3n+2}, Tx_{3n+3}, \ldots\}, \tag{23}$$

which is a Cauchy sequence in G-complete G-metric space and therefore converges to a limit point w. But all subsequences of a convergent sequence converge; so, we have

$$\begin{aligned} \lim_{n\to\infty} Rx_{3n} &= \lim_{n\to\infty} Qx_{3n+1} = w, \\ \lim_{n\to\infty} Sx_{3n} &= \lim_{n\to\infty} Jx_{3n+1} = w, \\ \lim_{n\to\infty} Tx_{3n-1} &= \lim_{n\to\infty} Ix_{3n} = w. \end{aligned} \tag{24}$$

Since $\{S, I\}$ are weakly commuting mappings, thus we have

$$G(SIx_{3n}, ISx_{3n}, ISx_{3n}) \le G(Ix_{3n}, Sx_{3n}, Sx_{3n}). \tag{25}$$

Taking limit $n \to \infty$ and noting that S and I are continuous mappings, we have

$$G(Sw, Iw, Iw) \le G(w, w, w), \tag{26}$$

which gives the notion that $Sw = Iw$. Analogously, we can get $Tw = Jw$ and $Rw = Qw$. We claim that $Rw \neq Sw$ and $Sw \neq Tw$ and then from condition (3)

$$\begin{aligned} G(Rw, Sw, Tw) &\leq A \\ &\cdot \big[G(Rw, Sw, Tw)\, G(Rw, Sw, Sw) \\ &+ [G(Rw, Tw, Sw)]^2 \\ &+ G(Rw, Sw, Sw)\, G(Rw, Tw, Sw)\big] \\ &\cdot (G(Rw, Sw, Sw) + G(Rw, Tw, Sw) \\ &+ G(Rw, Sw, Sw))^{-1} + B \cdot G(Rw, Tw, Sw), \\ G(Rw, Sw, Tw) &\leq (A + B)\, G(Rw, Tw, Sw), \end{aligned} \tag{27}$$

which is a contraction:

$$G(Rw, Sw, Tw) = 0 \quad \text{implies } Rw = Sw = Tw. \tag{28}$$

Similarly, using similar arguments to those given above, we obtain a contradiction for $Rw \neq Sw$ and $Sw = Tw$ or for $Rw = Sw$ and $Sw \neq Tw$. Hence, in all the cases, we conclude that $Rw = Sw = Tw$. We prove that any fixed point of R is a fixed point of S, T, Q, I, and J. Assume that $w \in X$ is such that $Rw = w$. Now, we prove that $w = Tw = Sw$. If it is not the case, then, for $w \neq Sw$ and $w \neq Tw$, we get

$$\begin{aligned} G(w, Sw, Tw) &= G(Rw, Sw, Tw) \leq A \\ &\cdot \big[G(Rw, Sw, Tw)\, G(Rw, Sw, Sw) \\ &+ [G(Rw, Tw, Sw)]^2 \\ &+ G(Rw, Sw, Sw)\, G(Rw, Tw, Sw)\big] \\ &\cdot (G(Rw, Sw, Sw) + G(Rw, Tw, Sw) \\ &+ G(Rw, Sw, Sw))^{-1} + B \cdot G(Rw, Tw, Sw), \\ G(w, Sw, Tw) &\leq (A + B)\, G(w, Tw, Sw), \end{aligned} \tag{29}$$

where $G(w, Sw, Tw) = 0$ which implies that $w = Sw = Tw$; in a similar argument, we can prove the other cases.

Uniqueness. Suppose that S, T, R, I, J, and Q have two common fixed points z and w such that $z \neq w$. Since condition $G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix) = 0$ implies $G(Sx, Ty, Rz) + G(Qx, Jy, Iz) = 0$, we have that $G(Rz, Sz, Iz) + G(Qz, Jz, Iw) + G(Rz, Sz, Iz) = 0$ implies $G(Sz, Tz, Rw) + G(Qz, Jz, Iw) = 0$, which can be written as $G(Sz, Tz, Rw) = 0$ or $G(Qz, Jz, Iw) = 0$.

Therefore, one can get the following:

$$\begin{aligned} &G(z, z, w) = 0 \\ &\text{or } G(z, z, w) = 0 \quad \text{implies that } z = w. \end{aligned} \tag{30}$$

□

Theorem 13 produces the following corollaries.

Corollary 14. *Let (X, G) be a G-complete G-metric space and let $R, S, T, I, J, Q : X \to X$ be three self-maps and let $\{S, I\}$, $\{T, J\}$, and $\{R, Q\}$ be weakly commuting pairs of self-mapping such that $T(X) \subset I(X)$, $S(X) \subset J(X)$, and $R(X) \subset Q(X)$, satisfying*

$$G(Rx, Sy, Tz) \leq B \cdot G(Qx, Jy, Iz) \tag{31}$$

for all x, y, z in X with $x \neq y \neq z \neq x$ with $0 \leq B < 1$. Then, R, S, T, I, J, and Q have a unique common fixed point in X.

Proof. It follows by taking $A = 0$ in Theorem 13. □

Corollary 15. *Let (X, G) be a G-complete G-metric space and let $R, S, T, I, J, Q : X \to X$ be three self-maps and let $\{S, I\}$, $\{T, J\}$, and $\{R, Q\}$ be weakly commuting pairs of self-mapping such that $T(X) \subset I(X)$, $S(X) \subset J(X)$, and $R(X) \subset Q(X)$, satisfying*

$$\begin{aligned} G(Rx, Sy, Tz) &\leq A \cdot \big[G(Qx, Sx, Iz)\, G(Rx, Sx, Ix) \\ &+ [G(Qx, Jy, Iz)]^2 \\ &+ G(Rx, Sx, Ix)\, G(Qx, Jy, Iz)\big] (G(Rx, Sx, Ix) \\ &+ G(Qx, Jy, Iz) + G(Rx, Sx, Ix))^{-1} + B \\ &\cdot G(Rz, Tz, Sz) \end{aligned} \tag{32}$$

for all x, y, z in X with $x \neq y \neq z \neq x$ $A \geq 0$ with $0 \leq A < 1$, $G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix) \neq 0$. Then, R, S, T, I, J, and Q have a common fixed point. Further, if $G(Rx, Sx, Ix) + G(Qx, Jy, Iz) + G(Rx, Sx, Ix) = 0$ implies $G(Sx, Ty, Rz) + G(Qx, Jy, Iz) = 0$, then R, S, T, I, J, and Q have a unique common fixed point in X.

Proof. It follows by taking $B = 0$ in Theorem 13. □

Corollary 16. *Let (X, G) be a G-complete G-metric space and let $T, R, I, J : X \to X$ be three self-maps and let $\{T, I\}$, $\{T, J\}$, and $\{R, I\}$ be weakly commuting pairs of self-mapping such that $T(X) \subset I(X)$, $T(X) \subset J(X)$, and $R(X) \subset I(X)$, satisfying*

$$\begin{aligned} G(Rx, Ty, Tz) &\leq A \cdot \big[G(Ix, Tx, Iz)\, G(Rx, Tx, Ix) \\ &+ [G(Ix, Jy, Iz)]^2 + G(Rx, Tx, Ix)\, G(Ix, Jy, Iz)\big] \\ &\cdot (G(Rx, Tx, Ix) + G(Ix, Jy, Iz) \\ &+ G(Rx, Tx, Ix))^{-1} + B \cdot G(Ix, Jy, Iz) \end{aligned} \tag{33}$$

for all $x, y, z \in X$ with $x \neq y \neq z \neq x$ $A, B \geq 0$ with $0 \leq A + B < 1$, $G(Rx, Tx, Ix) + G(Ix, Jy, Iz) + G(Rx, Tx, Ix) \neq 0$. Then, T, R, I, and J have a common fixed point. Further, if $G(Rx, Tx, Ix) + G(Ix, Jy, Iz) + G(Rx, Tx, Ix) = 0$ implies $G(Sx, Ty, Rz) + G(Ix, Jy, Iz) = 0$, then T, R, I, and J have a unique common fixed point in X.

Proof. The proof follows by setting $S = T$ and $I = Q$ in Theorem 13. □

Competing Interests

The authors declare that they have no competing interests.

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