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Some coincidence point results for generalized (ψ,φ)-weakly contractions in ordered b-metric spaces

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Abstract
In this paper we present some coincidence point results for four mappings satisfying generalized (ψ,φ)-weakly contractive condition in the framework of ordered b-metric spaces. Our results extend, generalize, unify, enrich, and complement recently results of Nashine and Samet (Nonlinear Anal. 74:2201-2209, 2011) and Shatanawi and Samet (Comput. Math. Appl. 62:3204-3214, 2011). As an application of our results, periodic points of weakly contractive mappings are obtained. Also, an example is given to support our results.

MSC: 47H10; 54H25

Keywords: b-metric space; partially ordered set; fixed point; altering distance function

1 Introduction

A self-mapping f on a metric space (X,d) is a contraction, if $d(fx,fy) \le kd(x,y)$ for all $x,y \in X$, where $k \in [0,1)$.

The Banach contraction principle, which shows that every contractive mapping defined on a complete metric space has a unique fixed point, is one of the famous theorems which was generalized by many researchers in different ways [1–5] and [6–12].

A self-mapping f on X is a weak contraction, if $d(fx,fy) \le d(x,y) - \varphi(d(x,y))$ for all $x,y \in X$, where φ is an altering distance function.

The above concept was introduced by Alber and Guerre-Delabriere [13] in the setup of Hilbert spaces. Rhoades [14] generalized the Banach contraction principle by considering this class of mappings in the setup of metric spaces and proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point.

Let f and g be two self-mappings on a nonempty set X. If $x = fx = gx$ for some x in X, then x is called a common fixed point of f and g.

Zhang and Song [15] introduced the concept of a generalized φ-weak contractive mappings and proved the following common fixed point result.

Theorem 1 [15] *Let (X,d) be a complete metric space. If $f,g : X \to X$ are generalized φ-weak contractive mappings, then there exists a unique point $u \in X$ such that $u = fu = gu$.*

For further work in this direction, we refer to [1, 16, 17] and [18].

Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in this spaces. For more details of fixed point results, its applications, comparison of different contractive conditions, and related results in ordered metric spaces we refer the reader to [2, 5, 19–28] and the references mentioned therein.

The concept of a b-metric space was introduced by Czerwik in [29]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see also [30–41]).

In this paper, we prove some coincidence point results for nonlinear generalized (ψ,φ)-weakly contractive mappings in partially ordered b-metric spaces. Our results extend and generalize the results in [22] and [25] from the context of ordered metric spaces to the setting of ordered b-metric spaces.

2 Preliminaries

Definition 1 Let f and g be two self-maps on partially ordered set X. A pair (f,g) is said to be:

(i) weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$ [42],
(ii) partially weakly increasing if $fx \preceq gfx$ for all $x \in X$ [2].

Let X be a nonempty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 2 Let $(X,\preceq)$ be a partially ordered set and $f,g,h : X \to X$ are mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. The ordered pair (f,g) is said to be:

(a) weakly increasing with respect to h if and only if for all $x \in X$, $fx \preceq gy$ for all $y \in h^{-1}(fx)$ and $gx \preceq fy$ for all $y \in h^{-1}(gx)$ [22],
(b) partially weakly increasing with respect to h if $fx \preceq gy$ for all $y \in h^{-1}(fx)$ [20].

Remark 1 In the above definition: (i) if $f = g$, we say that f is weakly increasing (partially weakly increasing) with respect to h, (ii) if $h = I_X$ (the identity mapping on X), then the above definition reduces to the weakly increasing (partially weakly increasing) mapping (see [22, 25]).

Jungck in [43] introduced the following definition.

Definition 3 [43] Let (X,d) be a metric space and $f,g : X \to X$. The pair (f,g) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 4 [44] Let $f,g : X \to X$ be given self-mappings on X. The pair (f,g) is said to be weakly compatible if f and g commute at their coincidence points (*i.e.*, $fgx = gfx$, whenever $fx = gx$).

Definition 5 Let $(X,\preceq)$ be a partially ordered set and d be a metric on X. We say that $(X,d,\preceq)$ is regular if the following conditions hold:

(i) if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
(ii) if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all n.

In [22], Nashine and Samet, by considering a pair of altering distance functions (ψ,φ), established some coincidence point and common fixed point theorems for mappings satisfying a generalized weakly contractive condition in an ordered complete metric space. They proved the following theorem.

Theorem 2 ([22], Theorem 2.4) *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T,R:X\to X$ be given mappings satisfying for every pair $(x,y)\in X\times X$ such that Rx and Ry are comparable,*

$$\psi\bigl(d(Tx,Ty)\bigr)\le\psi\bigl(d(Rx,Ry)\bigr)-\varphi\bigl(d(Rx,Ry)\bigr),$$

where ψ and φ are altering distance functions. We suppose the following hypotheses to hold:

(i) *T and R are continuous,*

(ii) *$TX\subseteq RX$,*

(iii) *T is weakly increasing with respect to R,*

(iv) *the pair (T,R) is compatible.*

Then T and R have a coincidence point, that is, there exists $u\in X$ such that $Ru=Tu$.

Also, they showed that by replacing the continuity hypotheses on T and R with the regularity of $(X,d,\preceq)$ and omitting the compatibility of the pair (T,R), the above theorem is still valid (see Theorem 2.6 of [22]).

Also, in [25], Shatanawi and Samet studied common fixed point and coincidence point for three self-mappings T, S, and R satisfying (ψ,φ)-weakly contractive condition in an ordered metric space (X,d), where S and T are weakly increasing with respect to R and ψ, φ are altering distance functions. Their result generalizes Theorem 2.

Shatanawi and Samet proved the following result.

Theorem 3 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $T,S,R:X\to X$ be three mappings such that for all $x,y\in X$ for which Rx and Ry are comparable, we have*

$$\psi\bigl(d(Tx,Sy)\bigr)\le\psi\bigl(M(x,y)\bigr)-\phi\bigl(M(x,y)\bigr),$$

where

$$M(x,y)\in\left\{d(Rx,Ry),\frac{d(Rx,Tx)+d(Ry,Sy)}{2},\frac{d(Tx,Ry)+d(Rx,Sy)}{2}\right\}$$

and ψ and ϕ are altering distance functions. Assume that T, S, and R satisfy the following hypotheses:

(i) *T and S are weakly increasing with respect to R,*

(ii) *$TX\subseteq RX$, $SX\subseteq RX$, and R is continuous.*

Let either

(iii) *the pair (T,R) is compatible and T is continuous, or*

(iv) *the pair (S,R) is compatible and S is continuous.*

Then T, S, and R have a coincidence point, that is, there exists $u\in X$ such that $Ru=Tu=Su$.

Analogous to the work in [22], Shatanawi and Samet proved the above result by replacing the continuity hypotheses of T, S, and R with the regularity of X and omitting the compatibility of the pair (T,R) and (S,R) (see Theorem 2.2 of [25]).

In [45], Radenović *et al.* studied common fixed point for two mappings satisfying (ψ,φ)-weakly contractive condition, but without order. The difference is that they do not use the maximum of the set, but its arbitrary element.

Consistent with [29, 46] and [40], the following definitions and results will be needed in the sequel.

Definition 6 [29] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric iff, for all $x,y,z \in X$, the following conditions are satisfied:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than the class of metric spaces, since a b-metric is a metric, when $s = 1$.

The following example shows that in general a b-metric need not necessarily be a metric. (see, also, [40], p.264).

Example 1 [47] Let (X,d) be a metric space, and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

However, if (X,d) is a metric space, then (X,ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x,y) = |x-y|$ is the usual Euclidean metric, then $\rho(x,y) = (x-y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but not a metric on $\mathbb{R}$.

The following example of a b-metric space is given in [48].

Example 2 [48] Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |f(x)|^2\,dx < \infty$. Define $D : X \times X \to [0,\infty)$ by $D(f,g) = \int_0^1 |f(x) - g(x)|^2\,dx$. As $(\int_0^1 |f(x) - g(x)|^2\,dx)^{\frac{1}{2}}$ is a metric on X, from the previous example, D is a b-metric on X, with $s = 2$.

Khamsi [49] also showed that each cone metric space over a normal cone has a b-metric structure.

We also need the following definitions.

Definition 7 Let X be a nonempty set. Then $(X,d,\preceq)$ is called a partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X,\preceq)$.

Definition 8 [34] Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) b-convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$, as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(b) b-Cauchy if and only if $d(x_n,x_m) \to 0$, as $n,m \to +\infty$.

Proposition 1 (See Remark 2.1 in [34]) *In a b-metric space* (X,d) *the following assertions hold*:

(p_1) *A b-convergent sequence has a unique limit.*
(p_2) *Each b-convergent sequence is b-Cauchy.*
(p_3) *In general, a b-metric is not continuous.*

Also, recently, Hussain *et al.* have presented an example of a b-metric which is not continuous (see Example 3 in [36]).

Definition 9 [34] The b-metric space (X,d) is b-complete if every b-Cauchy sequence in X b-converges.

Definition 10 [34] Let (X,d) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all b-convergent sequences of points in Y, *i.e.*,

$$\overline{Y} = \left\{x \in X: \text{ there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } \lim_{n\to\infty} x_n = x\right\}.$$

Taking into account the above definition, we have the following concepts.

Definition 11 [34] Let (X,d) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y, which b-converges to an element x, we have $x \in Y$ (*i.e.*, $\overline{Y} = Y$).

Definition 12 Let (X,d) and (X',d') be two b-metric spaces. Then a function $f: X \to X'$ is b-continuous at a point $x \in X$ if and only if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f(x_n)\}$ is b-convergent to $f(x)$.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences.

Lemma 1 [47] *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq s d(x,z).$$

Motivated by the work in [20, 22] and [25], we prove some coincidence point results for nonlinear generalized (ψ,φ)-weakly contractive mappings in partially ordered b-metric spaces. Our results extend and generalize the results in [22] and [25] from the context of ordered metric spaces to the setting of ordered b-metric spaces.

3 Main results

Let $(X,\preceq,d)$ be an ordered b-metric space and $f,g,R,S: X \to X$ be four self-mappings. Throughout this paper, unless otherwise stated, let

$$M(x,y) \in \left\{d(Sx,Ry), \frac{d(Sx,fx) + d(Ry,gy)}{2s}, \frac{d(Sx,gy) + d(Ry,fx)}{2s^2}\right\}$$

for all $x,y \in X$.

Theorem 4 *Let $(X,\preceq,d)$ be an ordered complete b-metric space. Let $f,g,R,S:X\to X$ be four mappings such that $f(X)\subseteq R(X)$ and $g(X)\subseteq S(X)$. Suppose that for every $x,y\in X$ with comparable elements Sx, Ry, there exists $M(x,y)$ such that*

$$\psi\bigl(s^4 d(fx,gy)\bigr)\le\psi\bigl(M(x,y)\bigr)-\varphi\bigl(M(x,y)\bigr),\tag{3.1}$$

where $\psi,\varphi:[0,\infty)\to[0,\infty)$ are altering distance functions. Let f, g, R, and S are continuous, the pairs (f,S) and (g,R) are compatible and the pairs (f,g) and (g,f) are partially weakly increasing with respect to R and S, respectively. Then the pairs (f,S) and (g,R) have a coincidence point z in X. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R, and S.

Proof Let x_0 be an arbitrary point of X. Choose $x_1\in X$ such that $fx_0=Rx_1$ and $x_2\in X$ such that $gx_1=Sx_2$. This can be done as $f(X)\subseteq R(X)$ and $g(X)\subseteq S(X)$.

Continuing this way, construct a sequence $\{z_n\}$ defined by

$$z_{2n+1}=Rx_{2n+1}=fx_{2n}$$

and

$$z_{2n+2}=Sx_{2n+2}=gx_{2n+1}$$

for all $n\ge 0$.

As $x_1\in R^{-1}(fx_0)$ and $x_2\in S^{-1}(gx_1)$, and the pairs (f,g) and (g,f) are partially weakly increasing with respect to R and S, respectively, we have

$$Rx_1=fx_0\preceq gx_1=Sx_2\preceq fx_2=Rx_3.$$

Repeating this process, we obtain $z_{2n+1}\preceq z_{2n+2}$ for all $n\ge 0$.

We will complete the proof in three steps.

Step I. We will prove that $\lim_{k\to\infty}d(z_k,z_{k+1})=0$.

Define $d_k=d(z_k,z_{k+1})$. Suppose $d_{k_0}=0$ for some k_0. Then $z_{k_0}=z_{k_0+1}$. In the case that $k_0=2n$, then $z_{2n}=z_{2n+1}$ gives $z_{2n+1}=z_{2n+2}$. Indeed,

$$\begin{aligned}\psi\bigl(s^4 d(z_{2n+1},z_{2n+2})\bigr)&=\psi\bigl(s^3 d(fx_{2n},gx_{2n+1})\bigr)\\&\le\psi\bigl(M(x_{2n},x_{2n+1})\bigr)-\varphi\bigl(M(x_{2n},x_{2n+1})\bigr),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1})\in{}&\biggl\{d(Sx_{2n},Rx_{2n+1}),\\&\frac{d(Sx_{2n},fx_{2n})+d(Rx_{2n+1},gx_{2n+1})}{2s},\\&\frac{d(Sx_{2n},gx_{2n+1})+d(Rx_{2n+1},fx_{2n})}{2s^2}\biggr\}\\={}&\biggl\{d(z_{2n},z_{2n+1}),\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s},\end{aligned}$$

$$\left.\frac{d(z_{2n},z_{2n+2})+d(z_{2n+1},z_{2n+1})}{2s^2}\right\}$$

$$=\left\{0,\frac{d(z_{2n+1},z_{2n+2})}{2s},\frac{d(z_{2n},z_{2n+2})}{2s^2}\right\}.$$

Taking $M(x_{2n},x_{2n+1})=\frac{d(z_{2n+1},z_{2n+2})}{2s}$, then from (3.2) we have

$$\begin{aligned}\psi\bigl(s^4d(z_{2n+1},z_{2n+2})\bigr)&\le\psi\left(\frac{d(z_{2n+1},z_{2n+2})}{2s}\right)-\varphi\left(\frac{d(z_{2n+1},z_{2n+2})}{2s}\right)\\&\le\psi\bigl(d(z_{2n+1},z_{2n+2})\bigr)-\varphi\left(\frac{d(z_{2n+1},z_{2n+2})}{2s}\right)\\&\le\psi\bigl(s^3d(z_{2n+1},z_{2n+2})\bigr)-\varphi\left(\frac{d(z_{2n+1},z_{2n+2})}{2s}\right),\end{aligned}\tag{3.3}$$

which implies that $\varphi(\frac{d(z_{2n+1},z_{2n+2})}{2s})=0$, that is, $z_{2n}=z_{2n+1}=z_{2n+2}$. Similarly, if $k_0=2n+1$, then $z_{2n+1}=z_{2n+2}$ gives $z_{2n+2}=z_{2n+3}$. Consequently, the sequence $\{z_k\}$ becomes constant for $k\ge k_0$ and z_{k_0} is a coincidence point of the pairs (f,S) and (g,R). To this aim, let $k_0=2n$. Since $z_{2n}=z_{2n+1}=z_{2n+2}$,

$$z_{2n}=Sx_{2n}=z_{2n+1}=Rx_{2n+1}=fx_{2n}=z_{2n+2}=gx_{2n+1}=Sx_{2n+2}.$$

This means that $S(x_{2n})=f(x_{2n})$ and $R(x_{2n+1})=g(x_{2n+1})$.

On the other hand, the pairs (f,S) and (g,R) are compatible. So, they are weakly compatible. Hence, $fS(x_{2n})=Sf(x_{2n})$ and $gR(x_{2n+1})=Rg(x_{2n+1})$, or, equivalently, $fz_{2n}=Sz_{2n+1}$ and $gz_{2n+1}=Rz_{2n+2}$. Now, since $z_{2n}=z_{2n+1}=z_{2n+2}$, we have $fz_{2n}=Sz_{2n}$ and $gz_{2n}=Rz_{2n}$.

In the other case, when $k_0=2n+1$, similarly, one can show that z_{2n+1} is a coincidence point of the pairs (f,S) and (g,R).

Note that, when $M(x_{2n},x_{2n+1})=0$ or, $M(x_{2n},x_{2n+1})=\frac{d(z_{2n},z_{2n+2})}{2s^2}$, the desired result is obtained.

Now, suppose that

$$d_k=d(z_k,z_{k+1})>0\tag{3.4}$$

for each k. We claim that

$$d(z_{k+1},z_{k+2})\le d(z_k,z_{k+1})\tag{3.5}$$

for each $k=1,2,3,\ldots$.

Let $k=2n$ and, for $n\ge0$, $d(z_{2n+1},z_{2n+2})\ge d(z_{2n},z_{2n+1})>0$. Then, as $Sx_{2n}\preceq Rx_{2n+1}$, using (3.1) we obtain

$$\begin{aligned}\psi\bigl(s^4d(z_{2n+1},z_{2n+2})\bigr)&=\psi\bigl(s^3d(fx_{2n},gx_{2n+1})\bigr)\\&\le\psi\bigl(M(x_{2n},x_{2n+1})\bigr)-\varphi\bigl(M(x_{2n},x_{2n+1})\bigr),\end{aligned}\tag{3.6}$$

where

$$M(x_{2n},x_{2n+1}) \in \left\{d(Sx_{2n},Rx_{2n+1}), \frac{d(Sx_{2n},fx_{2n})+d(Rx_{2n+1},gx_{2n+1})}{2s}, \frac{d(Sx_{2n},gx_{2n+1})+d(Rx_{2n+1},fx_{2n})}{2s^2}\right\}$$
$$= \left\{d(z_{2n},z_{2n+1}), \frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s}, \frac{d(z_{2n},z_{2n+2})+d(z_{2n+1},z_{2n+1})}{2s^2}\right\}.$$

If

$$M(x_{2n},x_{2n+1}) = \frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s} \le \frac{d(z_{2n+1},z_{2n+2})}{s},$$

as $d(z_{2n+1},z_{2n+2}) \ge d(z_{2n},z_{2n+1})$, then from (3.6), we have

$$\begin{aligned}\psi\big(s^4 d(z_{2n+1},z_{2n+2})\big) &\le \psi\left(\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s}\right) \\ &\quad - \varphi\left(\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s}\right) \\ &\le \psi\big(s^3 d(z_{2n+1},z_{2n+2})\big) - \varphi\left(\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s}\right),\end{aligned} \quad (3.7)$$

which implies that

$$\varphi\left(\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s}\right) \le 0,$$

this is possible only if $\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2s} = 0$, that is, $d(z_{2n},z_{2n+1}) = 0$, a contradiction to (3.4). Hence, $d(z_{2n+1},z_{2n+2}) \le d(z_{2n},z_{2n+1})$, for all $n \ge 0$.

Therefore, (3.5) is proved for $k = 2n$.

Similarly, it can be shown that

$$d(z_{2n+2},z_{2n+3}) \le d(z_{2n+1},z_{2n+2}) \quad (3.8)$$

for all $n \ge 0$.

Analogously, in all cases, we see that $\{d(z_k,z_{k+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an $r \ge 0$ such that

$$\lim_{k\to\infty} d(z_k,z_{k+1}) = r. \quad (3.9)$$

We know that

$$M(x_{2n},x_{2n+1}) \in \left\{d(Sx_{2n},Rx_{2n+1}), \frac{d(Sx_{2n},fx_{2n})+d(Rx_{2n+1},gx_{2n+1})}{2s}, \frac{d(Sx_{2n},gx_{2n+1})+d(Rx_{2n+1},fx_{2n})}{2s^2}\right\}$$

$$= \left\{ d(z_{2n}, z_{2n+1}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}, \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s^2} \right\}.$$

Substituting the values of $M(x_{2n}, x_{2n+1})$ in (3.6) and then taking the limit as $n \to \infty$ in (3.6), we obtain $r = 0$. For instance, let

$$M(x_{2n}, x_{2n+1}) = \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s^2}.$$

So, we have

$$\begin{aligned} \psi\left(s^4 d(z_{2n+1}, z_{2n+2})\right) &\leq \psi\left(\frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s^2}\right) \\ &\quad - \varphi\left(\frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s^2}\right) \\ &= \psi\left(\frac{d(z_{2n}, z_{2n+2})}{2s^2}\right) - \varphi\left(\frac{d(z_{2n}, z_{2n+2})}{2s^2}\right) \\ &\leq \psi\left(\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2s}\right) - \varphi\left(\frac{d(z_{2n}, z_{2n+2})}{2s^2}\right). \end{aligned} \tag{3.10}$$

Letting $n \to \infty$ in (3.10), using (3.9) and the continuity of ψ and φ, we have

$$\varphi\left(\lim_{n\to\infty} \frac{d(z_{2n}, z_{2n+2})}{2s^2}\right) = 0.$$

Hence, $\lim_{n\to\infty} \frac{d(z_{2n}, z_{2n+2})}{2s^2} = 0$, from our assumptions as regards φ.

Now, taking into account (3.10) and letting $n \to \infty$, we find that $\psi(s^3 r) \leq \psi(0) - \varphi(0)$. Hence, $r = 0$. In general, for the other values of $M(x_{2n}, x_{2n+1})$ we can show that

$$r = \lim_{k\to\infty} d(z_k, z_{k+1}) = \lim_{n\to\infty} d(z_{2n}, z_{2n+1}) = 0. \tag{3.11}$$

Step II. We will show that $\{z_n\}$ is a b-Cauchy sequence in X. That is, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n \geq k$, $d(z_m, z_n) < \varepsilon$.

Assume to the contrary that there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{2m(k)}\}$ and $\{z_{2n(k)}\}$ of $\{z_{2n}\}$ such that $n(k) > m(k) \geq k$ and

$$d(z_{2m(k)}, z_{2n(k)}) \geq \varepsilon, \tag{3.12}$$

and $n(k)$ is the smallest number such that the above condition holds; *i.e.*,

$$d(z_{2m(k)}, z_{2n(k)-2}) < \varepsilon. \tag{3.13}$$

From the triangle inequality and (3.12) and (3.13), we have

$$\begin{aligned} \varepsilon &\leq d(z_{2m(k)}, z_{2n(k)}) \\ &\leq s\left[d(z_{2m(k)}, z_{2n(k)-2}) + d(z_{n(k)-2}, z_{2n(k)})\right] \\ &< s\varepsilon + s d(z_{2n(k)-2}, z_{2n(k)}). \end{aligned} \tag{3.14}$$

Taking the limit as $k \to \infty$ in (3.14), from (3.11) we obtain

$$\varepsilon \leq \limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)}) \leq s\varepsilon. \tag{3.15}$$

Using the triangle inequality again we have

$$\begin{aligned} d(z_{2m(k)}, z_{2n(k)}) &\leq s\bigl[d(z_{2m(k)}, z_{2n(k)+1}) + d(z_{2n(k)+1}, z_{2n(k)})\bigr] \\ &\leq s^2\bigl[d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)}, z_{2n(k)+1})\bigr] + sd(z_{2n(k)}, z_{2n(k)+1}). \end{aligned} \tag{3.16}$$

Taking the limit as $k \to \infty$ in (3.16) and using (3.11) and (3.15), we have

$$\varepsilon \leq s \limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)+1}) \leq s^3\varepsilon,$$

or, equivalently,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)+1}) \leq s^2\varepsilon. \tag{3.17}$$

Using the triangle inequality again we have

$$\begin{aligned} d(z_{2m(k)-1}, z_{2n(k)+1}) &\leq s\bigl[d(z_{2m(k)-1}, z_{2m(k)}) + d(z_{2m(k)}, z_{2n(k)+1})\bigr] \\ &\leq sd(z_{2m(k)-1}, z_{2m(k)}) + s^2\bigl[d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)}, z_{2n(k)+1})\bigr]. \end{aligned} \tag{3.18}$$

Letting $k \to \infty$ in the above inequality, we have

$$\limsup_{k\to\infty} d(z_{2m(k)-1}, z_{2n(k)+1}) \leq s^3\varepsilon. \tag{3.19}$$

Using the triangle inequality again we have

$$\begin{aligned} \varepsilon &\leq d(z_{2m(k)}, z_{2n(k)}) \leq s\bigl[d(z_{2m(k)}, z_{2m(k)-1}) + d(z_{2m(k)-1}, z_{2n(k)})\bigr] \\ &\leq sd(z_{2m(k)}, z_{2m(k)-1}) + s^2\bigl[d(z_{2m(k)-1}, z_{2n(k)+1}) + d(z_{2n(k)+1}, z_{2n(k)})\bigr]. \end{aligned} \tag{3.20}$$

Letting $k \to \infty$ in the above inequality, we have

$$\limsup_{k\to\infty} d(z_{2m(k)-1}, z_{2n(k)+1}) \geq \frac{\varepsilon}{s^2}. \tag{3.21}$$

Also,

$$\begin{aligned} &d(z_{2m(k)}, z_{2n(k)}) \\ &\quad\leq s\bigl[d(z_{2m(k)}, z_{2m(k)+1}) + d(z_{2m(k)+1}, z_{2n(k)})\bigr] \\ &\quad\leq sd(z_{2m(k)}, z_{2m(k)+1}) + s^2\bigl[d(z_{2m(k)+1}, z_{2n(k)+2}) + d(z_{2n(k)+2}, z_{2n(k)})\bigr]. \end{aligned} \tag{3.22}$$

Letting $k \to \infty$ and using (3.11) and (3.15), we have

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)+2}). \tag{3.23}$$

As $Sx_{2m(k)} \preceq Rx_{2n(k)+1}$, from (3.1), we have

$$\psi\bigl(s^4 d(z_{2m(k)+1}, z_{2n(k)+2})\bigr) = \psi\bigl(s^3 d(fx_{2m(k)}, gx_{2n(k)+1})\bigr) \leq \psi\bigl(M(x_{2m(k)}, x_{2n(k)+1})\bigr) - \varphi\bigl(M(x_{2m(k)}, x_{2n(k)+1})\bigr), \tag{3.24}$$

where

$$\begin{aligned} M(x_{2m(k)}, x_{2n(k)+1}) &\in \biggl\{ d(Sx_{2m(k)}, Rx_{2n(k)+1}), \frac{d(Sx_{2m(k)}, fx_{2m(k)}) + d(Rx_{2n(k)+1}, gx_{2n(k)+1})}{2s}, \\ &\quad \frac{d(Sx_{2m(k)}, gx_{2n(k)+1}) + d(Rx_{2n(k)+1}, fx_{2m(k)})}{2s^2} \biggr\} \\ &= \biggl\{ d(z_{2m(k)}, z_{2n(k)+1}), \frac{d(z_{2m(k)}, z_{2m(k)-1}) + d(z_{2n(k)+1}, z_{2n(k)})}{2s}, \\ &\quad \frac{d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)+1}, z_{2m(k)-1})}{2s^2} \biggr\}. \end{aligned}$$

If

$$M(x_{2m(k)}, x_{2n(k)+1}) = \frac{d(z_{2m(k)}, z_{2m(k)-1}) + d(z_{2n(k)+1}, z_{2n(k)})}{2s},$$

from (3.11), we get $\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}) = 0$. Hence, according to (3.24) we have $\lim_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)+2}) = 0$, which contradicts (3.23).

If

$$M(x_{2m(k)}, x_{2n(k)+1}) = \frac{d(z_{2m(k)}, z_{2n(k)}) + d(z_{2n(k)+1}, z_{2m(k)-1})}{2s^2},$$

from (3.15), (3.19), and (3.21), we get

$$\frac{\varepsilon}{2s^4} \leq \liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}) \leq \limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}) \leq s\varepsilon.$$

Taking the limit as $k \to \infty$ in (3.24), we have

$$\begin{aligned} \psi(s \cdot \varepsilon) &\leq \psi\left(s^4 \cdot \frac{\varepsilon}{s^2}\right) \\ &\leq \psi\Bigl(s^3 \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+2})\Bigr) \\ &\leq \psi\Bigl(\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1})\Bigr) - \varphi\Bigl(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1})\Bigr) \\ &\leq \psi(s\varepsilon) - \varphi\left(\frac{\varepsilon}{2s^4}\right), \end{aligned} \tag{3.25}$$

which implies that $\varphi(\frac{\varepsilon}{2s^4}) \leq 0$, hence, $\varepsilon = 0$, a contradiction.

If

$$M(x_{2m(k)}, x_{2n(k)+1}) = d(z_{2m(k)}, z_{2n(k)+1}),$$

from (3.17), by taking the limit as $k \to \infty$ in (3.24), we have

$$\begin{aligned}\psi\left(s^2 \cdot \varepsilon\right) &= \psi\left(s^4 \cdot \frac{\varepsilon}{s^2}\right) \\ &\le \psi\left(s^4 \limsup_{k\to\infty} d(z_{2m(k)+1}, z_{2n(k)+2})\right) \\ &\le \psi\left(\limsup_{k\to\infty} d(z_{2m(k)}, z_{2n(k)+1})\right) - \varphi\left(\liminf_{k\to\infty} d(z_{2m(k)}, z_{2n(k)+1})\right) \\ &\le \psi\left(s^2\varepsilon\right) - \varphi\left(\frac{\varepsilon}{s}\right),\end{aligned}$$

which implies that $\varphi(\frac{\varepsilon}{s}) \le 0$, hence, $\varepsilon = 0$, a contradiction.

Hence, $\{z_n\}$ is a b-Cauchy sequence.

Step III. We will show that f, g, R, and S have a coincidence point.

Since $\{z_n\}$ is a b-Cauchy sequence in the complete b-metric space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(z_{2n+1}, z) = \lim_{n\to\infty} d(Rx_{2n+1}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = 0 \tag{3.26}$$

and

$$\lim_{n\to\infty} d(z_{2n+2}, z) = \lim_{n\to\infty} d(Sx_{2n+2}, z) = \lim_{n\to\infty} d(gx_{2n+1}, z) = 0. \tag{3.27}$$

Hence,

$$Sx_{2n} \to z \quad \text{and} \quad fx_{2n} \to z, \quad \text{as } n \to \infty. \tag{3.28}$$

As (f, S) is compatible, so,

$$\lim_{n\to\infty} d(Sfx_{2n}, fSx_{2n}) = 0. \tag{3.29}$$

Moreover, from $\lim_{n\to\infty} d(fx_{2n}, z) = 0$, $\lim_{n\to\infty} d(Sx_{2n}, z) = 0$, and the continuity of S and f, we obtain,

$$\lim_{n\to\infty} d(Sfx_{2n}, Sz) = 0 = \lim_{n\to\infty} d(fSx_{2n}, fz). \tag{3.30}$$

By the triangle inequality, we have

$$\begin{aligned}d(Sz, fz) &\le s\left[d(Sz, Sfx_{2n}) + d(Sfx_{2n}, fz)\right] \\ &\le sd(Sz, Sfx_{2n}) + s^2\left[d(Sfx_{2n}, fSx_{2n}) + d(fSx_{2n}, fz)\right].\end{aligned} \tag{3.31}$$

Taking the limit as $n \to \infty$ in (3.31), we obtain

$$d(Sz, fz) \le 0,$$

which yields $fz = Sz$, that is, z is a coincidence point of f and S.

Similarly, it can be proved that $gz = Rz$. Now, let Rz and Sz be comparable. By (3.1) we have

$$\psi\big(s^4 d(fz,gz)\big) \le \psi\big(M(z,z)\big) - \varphi\big(M(z,z)\big), \tag{3.32}$$

where

$$\begin{aligned} M(z,z) &\in \left\{ d(Sz,Rz), \frac{d(Sz,fz)+d(Rz,gz)}{2s}, \frac{d(Sz,gz)+d(Rz,fz)}{2s^2} \right\} \\ &= \left\{ d(fz,gz), 0, \frac{d(fz,gz)}{s^2} \right\}. \end{aligned}$$

In all three cases (3.32) yields $fz = gz = Sz = Rz$. □

In the following theorem, we omit the continuity assumption of f, g, R, and S, and replace the compatibility of the pairs (f,S) and (g,R) by weak compatibility of the pairs.

Theorem 5 *Let $(X,\preceq,d)$ be a regular partially ordered b-metric space, $f,g,R,S: X\to X$ be four mappings such that $f(X)\subseteq R(X)$ and $g(X)\subseteq S(X)$ and RX and SX are complete subsets of X. Suppose that for comparable elements $Sx, Ry \in X$, we have*

$$\psi\big(s^4 d(fx,gy)\big) \le \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big), \tag{3.33}$$

where $\psi,\varphi:[0,\infty)\to[0,\infty)$ are altering distance functions. Then the pairs (f,S) and (g,R) have a coincidence point z in X provided that the pairs (f,S) and (g,R) are weakly compatible and the pairs (f,g) and (g,f) are partially weakly increasing with respect to R and S, respectively. Moreover, if Rz and Sz are comparable, then $z\in X$ is a coincidence point of f, g, R, and S.

Proof Following the proof of Theorem 4, there exists $z \in X$ such that

$$\lim_{k\to\infty} d(z_k,z) = 0. \tag{3.34}$$

Since $R(X)$ is complete and $\{z_{2n+1}\}\subseteq R(X)$, therefore $z\in R(X)$. Hence, there exists $u\in X$ such that $z = Ru$ and

$$\lim_{n\to\infty} d(z_{2n+1},Ru) = \lim_{n\to\infty} d(Rx_{2n+1},Ru) = 0. \tag{3.35}$$

Similarly, there exists $v \in X$ such that $z = Ru = Sv$ and

$$\lim_{n\to\infty} d(z_{2n},Sv) = \lim_{n\to\infty} d(Sx_{2n},Sv) = 0. \tag{3.36}$$

We prove that v is a coincidence point of f and S.

Since $Rx_{2n+1}\to z = Sv$, as $n\to\infty$, from regularity of X, $Rx_{2n+1}\preceq Sv$. Therefore, from (3.33), we have

$$\psi\big(s^4 d(fv,gx_{2n+1})\big) \le \psi\big(M(v,x_{2n+1})\big) - \varphi\big(M(v,x_{2n+1})\big), \tag{3.37}$$

where, from Lemma 1,

$$\begin{aligned} M(v,x_{2n+1}) &\in \biggl\{ d(Sv,Rx_{2n+1}), \frac{d(Sv,fv)+d(Rx_{2n+1},gx_{2n+1})}{2s}, \\ &\quad \frac{d(Sv,gx_{2n+1})+d(Rx_{2n+1},fv)}{2s^2} \biggr\} \\ &\to \biggl\{0, \frac{d(z,fv)}{2s}, \frac{d(z,fv)}{2s^2}\biggr\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in (3.37), using Lemma 1 and the continuity of ψ and φ, we can obtain $fv = z = Sv$.

As f and S are weakly compatible, we have $fz = fSv = Sfv = Sz$. Thus, z is a coincidence point of f and S.

Similarly it can be shown that z is a coincidence point of the pair (g,R).

The remaining part of the proof is done via similar arguments to Theorem 4. □

Taking $S = R$ in Theorem 4, we obtain the following result.

Corollary 1 *Let $(X,\preceq,d)$ be a partially ordered complete b-metric space and $f,g,R : X \to X$ be three mappings such that $f(X) \cup g(X) \subseteq R(X)$ and R is continuous. Suppose that for every $x,y \in X$ with comparable elements Rx, Ry, we have*

$$\psi\bigl(s^4 d(fx,gy)\bigr) \le \psi\bigl(M(x,y)\bigr) - \varphi\bigl(M(x,y)\bigr), \tag{3.38}$$

where

$$M(x,y) \in \biggl\{ d(Rx,Ry), \frac{d(Rx,fx)+d(Ry,gy)}{2s}, \frac{d(Rx,gy)+d(Ry,fx)}{2s^2} \biggr\}$$

and $\psi,\varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then f, g, and R have a coincidence point in X provided that the pair (f,g) is weakly increasing with respect to R and either

(a) *the pair (f,R) is compatible and f is continuous, or*

(b) *the pair (g,R) is compatible and g is continuous.*

Taking $R = S$ and $f = g$ in Theorem 4, we obtain the following coincidence point result.

Corollary 2 *Let $(X,\preceq,d)$ be a partially ordered complete b-metric space and $f,R : X \to X$ be two mappings such that $f(X) \subseteq R(X)$. Suppose that for every $x,y \in X$ for which Rx, Ry are comparable, we have*

$$\psi\bigl(s^4 d(fx,fy)\bigr) \le \psi\bigl(M(x,y)\bigr) - \varphi\bigl(M(x,y)\bigr), \tag{3.39}$$

where

$$M(x,y) \in \biggl\{ d(Rx,Ry), \frac{d(Rx,fx)+d(Ry,fy)}{2s}, \frac{d(Rx,fy)+d(Ry,fx)}{2s^2} \biggr\}$$

and $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then the pair (f,R) has a coincidence point in X provided that f and R are continuous, the pair (f,R) is compatible, and f is weakly increasing with respect to R.

Example 3 Let $X = [0,\infty)$ and d on X be given by $d(x,y) = |x-y|^2$, for all $x,y \in X$. We define an ordering '$\preceq$' on X as follows:

$$x \preceq y \iff y \le x, \quad \forall x,y \in X.$$

Define self-maps f, g, S, and R on X by

$$fx = \sinh^{-1} x, \qquad Rx = \sinh 3x,$$
$$gx = \sinh^{-1}\left(\frac{x}{2}\right), \qquad Sx = \sinh 6x.$$

To prove that (f,g) is partially weakly increasing with respect to R, let $x,y \in X$ be such that $y \in R^{-1}fx$, that is, $Ry = fx$. By the definition of f and R, we have $\sinh^{-1} x = \sinh 3y$ and $y = \frac{\sinh^{-1}(\sinh^{-1} x)}{3}$. As $\sinh x \ge (\sinh^{-1} x)$, for all $x \in X$, therefore $6x \ge \sinh^{-1}(\sinh^{-1} x)$, or,

$$fx = \sinh^{-1} x \ge \sinh^{-1}\left(\frac{1}{6}\sinh^{-1}(\sinh^{-1} x)\right) = \sinh^{-1}\left(\frac{1}{2}y\right) = gy.$$

Therefore, $fx \preceq gy$. Hence (f,g) is partially weakly increasing with respect to R.

To prove that (g,f) is partially weakly increasing with respect to S, let $x,y \in X$ be such that $y \in S^{-1}gx$. This means that $Sy = gx$. Hence, we have $\sinh^{-1}\frac{x}{2} = \sinh 6y$ and so, $y = \frac{\sinh^{-1}(\sinh^{-1}\frac{x}{2})}{6}$. As $\sinh x \ge (\sinh^{-1} x)$, for all $x \in X$, therefore $3x \ge \frac{x}{2} \ge \sinh^{-1}(\sinh^{-1}\frac{x}{2})$, or, $\frac{x}{2} \ge \frac{\sinh^{-1}(\sinh^{-1}\frac{x}{2})}{6}$, so,

$$gx = \sinh^{-1}\frac{x}{2} \ge \sinh^{-1}\left(\frac{1}{6}\sinh^{-1}\left(\sinh^{-1}\frac{x}{2}\right)\right) = \sinh^{-1}(y) = fy.$$

Therefore, $gx \preceq fy$.

Furthermore, $fX = gX = SX = RX = [0,\infty)$ and the pairs (f,S) and (g,R) are compatible. Indeed, let $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(t,fx_n) = \lim_{n\to\infty} d(t,Sx_n) = 0$, for some $t \in X$. Therefore, we have

$$\lim_{n\to\infty} \left|\sinh^{-1} x_n - t\right| = \lim_{n\to\infty} |\sinh 6x_n - t| = 0.$$

Continuity of $\sinh^{-1} x$ and $\sinh 6x$ on X implies that

$$\lim_{n\to\infty} |x_n - \sinh t| = \lim_{n\to\infty} \left|x_n - \frac{\sinh^{-1} t}{6}\right| = 0,$$

and the uniqueness of the limit gives $\sinh t = \frac{\sinh^{-1} t}{6}$. But,

$$\sinh t = \frac{\sinh^{-1} t}{6} \iff t = 0.$$

So, we have $t = 0$. Since f and S are continuous, we have

$$\lim_{n\to\infty} d(fSx_n, Sfx_n) = \lim_{n\to\infty} |fSx_n - Sfx_n|^2 = 0.$$

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ as $\psi(t) = bt$ and $\varphi(t) = (b-1)t$ for all $t \in [0,\infty)$, where $1 < b \leq \frac{36}{16}$.

Using the mean value theorem for the functions $\sinh^{-1} x$ and $\sinh x$ on the intervals $[x, \frac{y}{2}] \subset X$ and $[6x, 3y] \subset X$, respectively, we have

$$\begin{aligned}
\psi\bigl(2^4 d(fx,gy)\bigr) &= 16b|fx-gy|^2 = 16b\left|\sinh^{-1} x - \sinh^{-1}\left(\frac{y}{2}\right)\right|^2 \\
&\leq 16b\left|x - \frac{y}{2}\right|^2 \leq 16b\frac{|6x-3y|^2}{36} \\
&\leq \frac{16b}{36}|\sinh 6x - \sinh 3y|^2 \leq |Sx - Ry|^2 \\
&= d(Sx,Ry) = \psi\bigl(d(Sx,Ry)\bigr) - \varphi\bigl(d(Sx,Ry)\bigr).
\end{aligned}$$

Thus, (3.1) is satisfied for all $x, y \in X$ and $M(x,y) = d(Sx,Ry)$. Therefore, all the conditions of Theorem 4 are satisfied. Moreover, 0 is a coincidence point of f, g, R, and S.

Corollary 3 *Let $(X,\preceq,d)$ be a regular partially ordered b-metric space, $f,g,R : X \to X$ be three mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq R(X)$ and RX is a complete subset of X. Suppose that for comparable elements $Rx, Ry \in X$, we have*

$$\psi\bigl(s^4 d(fx,gy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \varphi\bigl(M(x,y)\bigr), \tag{3.40}$$

where

$$M(x,y) \in \left\{d(Rx,Ry), \frac{d(Rx,fx)+d(Ry,gy)}{2s}, \frac{d(Rx,gy)+d(Ry,fx)}{2s^2}\right\}$$

and $\psi,\varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then the pairs (f,R) and (g,R) have a coincidence point z in X provided that the pair (f,g) is weakly increasing with respect to R.

Corollary 4 *Let $(X,\preceq,d)$ be a regular partially ordered b-metric space, $f,R : X \to X$ be two mappings such that $f(X) \subseteq R(X)$ and RX is a complete subset of X. Suppose that for comparable elements $Rx, Ry \in X$, we have*

$$\psi\bigl(s^4 d(fx,fy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \varphi\bigl(M(x,y)\bigr), \tag{3.41}$$

where

$$M(x,y) \in \left\{d(Rx,Ry), \frac{d(Rx,fx)+d(Ry,fy)}{2s}, \frac{d(Rx,fy)+d(Ry,fx)}{2s^2}\right\}$$

and $\psi,\varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then the pair (f,S) have a coincidence point z in X provided that f is weakly increasing with respect to R.

Taking $R = S = I_X$ (the identity mapping on X) in Theorems 4 and 5, we obtain the following common fixed point result.

Corollary 5 *Let $(X, \preceq, d)$ be a partially ordered complete b-metric space. Let $f, g : X \to X$ be two mappings. Suppose that for every comparable elements $x, y \in X$,*

$$\psi\big(s^4 d(fx, gy)\big) \le \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big), \tag{3.42}$$

where

$$M(x,y) \in \left\{d(x,y), \frac{d(x,fx) + d(y,gy)}{2s}, \frac{d(x,gy) + d(y,fx)}{2s^2}\right\}$$

and $\psi, \varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then the pair (f,g) have a common fixed point z in X provided that the pair (f,g) is weakly increasing and either

(a) *f or g is continuous, or*

(b) *X is regular.*

Remark 2

Theorem 2.1 of [25] is a special case of Corollary 1.
Theorem 2.2 of [25] is a special case of Corollary 3.
Corollary 2.1 of [25] is a special case of Corollary 5.
Corollary 2.2 of [25] is a special case of Corollary 5.
Theorem 2.4 of [22] is a special case of Corollary 2.
Theorem 2.6 of [22] is a special case of Corollary 4.
Corollary 2.7 of [22] is a special case of Corollary 1 with $R = I_X$.

4 Periodic point results

Let $F(f) = \{x \in X : fx = x\}$, be the fixed point set of f.

Clearly, a fixed point of f is also a fixed point of f^n for every $n \in \mathbb{N}$; that is, $F(f) \subset F(f^n)$. However, the converse is false. For example, the mapping $f : \mathbb{R} \to \mathbb{R}$, defined by $fx = \frac{1}{2} - x$ has the unique fixed point $\frac{1}{4}$, but every $x \in \mathbb{R}$ is a fixed point of f^2. If $F(f) = F(f^n)$ for every $n \in \mathbb{N}$, then f is said to have property P. For more details, we refer the reader to [50–52] and the references mentioned therein.

Taking $f = g$ and $\psi = I_{[0,\infty)}$ (the identity mapping on $[0,\infty)$) in Corollary 5, we obtain the following fixed point result.

Corollary 6 *Let $(X, \preceq, d)$ be a partially ordered complete b-metric space. Let $f : X \to X$ be a mapping. Suppose that for every comparable elements $x, y \in X$,*

$$s^4 d(fx, fy) \le M(x,y) - \varphi\big(M(x,y)\big), \tag{4.1}$$

where

$$M(x,y) \in \left\{d(x,y), \frac{d(x,fx) + d(y,fy)}{2s}, \frac{d(x,fy) + d(y,fx)}{2s^2}\right\}$$

and $\varphi : [0,\infty) \to [0,\infty)$ is an altering distance function. Then f has a fixed point if f is weakly increasing and either

(a) *f is continuous*, *or*
(b) *X is regular.*

Theorem 6 *Let X and f be as in Corollary* 6. *Then f has property P.*

Proof From Corollary 6, $F(f) \neq \emptyset$. Let $u \in F(f^n)$ for some $n > 1$. We will show that $u = fu$. We have $f^{n-1}u \preceq f^n u$, as f is weakly increasing. Using (3.42), we obtain

$$\begin{aligned} d(u,fu) &= d\big(f^n u, f^{n+1} u\big) \\ &= d\big(ff^{n-1}u, ff^n u\big) \\ &\leq \frac{M(f^{n-1}u,f^n u) - \varphi(M(f^{n-1}u,f^n u))}{s^4}, \end{aligned}$$

where

$$M\big(f^{n-1}u,f^n u\big) \in \bigg\{ d\big(f^{n-1}u,f^n u\big), \\ \frac{d(f^{n-1}u,f^n u) + d(f^n u,f^{n+1}u)}{2s}, \\ \frac{d(f^{n-1}u,f^{n+1}u) + d(f^n u,f^n u)}{2s^2} \bigg\}.$$

If $M(f^{n-1}u,f^n u) = d(f^{n-1}u,f^n u)$, then we have

$$d(u,fu) \leq \frac{d(f^{n-1}u,f^n u) - \varphi(d(f^{n-1}u,f^n u))}{s^4}.$$

Starting from $d(f^{n-1}u,f^n u)$, and repeating the above process, we get

$$\begin{aligned} d(u,fu) &\leq \frac{1}{s^4}\big[d\big(f^{n-1}u,f^n u\big) - \varphi\big(d\big(f^{n-1}u,f^n u\big)\big)\big] \\ &\leq \left[\frac{1}{s^4}\right]^2 \big[d\big(f^{n-2}u,f^{n-1}u\big) - \varphi\big(d\big(f^{n-2}u,f^{n-1}u\big)\big)\big] - \frac{1}{s^4}\varphi\big(d\big(f^{n-1}u,f^n u\big)\big) \\ &\cdots \\ &\leq \left[\frac{1}{s^4}\right]^n d(u,fu) - \sum_{i=0}^{n-1}\left[\frac{1}{s^4}\right]^{i+1} \varphi\big(d\big(f^{n-(i+1)}u,f^{n-(i)}u\big)\big) \\ &\leq d(u,fu) - \sum_{i=0}^{n-1}\left[\frac{1}{s^4}\right]^{i+1} \varphi\big(d\big(f^{n-(i+1)}u,f^{n-(i)}u\big)\big), \end{aligned}$$

which from our assumptions as regards φ implies that

$$d\big(f^{n-(i+1)}u,f^{n-(i)}u\big) = 0$$

for all $0 \leq i \leq n-1$. Now, taking $i = n-1$, we have $u = fu$.

Now, let

$$M\big(f^{n-1}u,f^n u\big) = \frac{d(f^{n-1}u,f^n u) + d(f^n u,f^{n+1}u)}{2s}.$$

Using (3.42) we have

$$\begin{aligned} d(u,fu) &= d\bigl(f^n u,f^{n+1}u\bigr) \\ &= d\bigl(ff^{n-1}u,ff^n u\bigr) \\ &\leq \frac{1}{s^4}\biggl[\frac{d(f^{n-1}u,f^n u)+d(f^n u,f^{n+1}u)}{2s}-\varphi\biggl(\frac{d(f^{n-1}u,f^n u)+d(f^n u,f^{n+1}u)}{2s}\biggr)\biggr], \end{aligned}$$

that is,

$$\begin{aligned} d(u,fu) &= d\bigl(f^n u,f^{n+1}u\bigr) \\ &\leq \frac{2s^5}{2s^5-1}\biggl[\frac{1}{2s^5}d\bigl(f^{n-1}u,f^n u\bigr)-\frac{1}{s^4}\varphi\biggl(\frac{d(f^{n-1}u,f^n u)+d(f^n u,f^{n+1}u)}{2s}\biggr)\biggr]. \end{aligned}$$

Repeating the above process, we get

$$\begin{aligned} &d\bigl(f^{n-1}u,f^n u\bigr) \\ &\quad\leq \frac{2s^5}{2s^5-1}\biggl[\frac{1}{2s^5}d\bigl(f^{n-2}u,f^{n-1}u\bigr)-\frac{1}{s^4}\varphi\biggl(\frac{d(f^{n-2}u,f^{n-1}u)+d(f^{n-1}u,f^n u)}{2s}\biggr)\biggr]. \end{aligned}$$

From the above inequalities, we have

$$\begin{aligned} d(u,fu) &\leq \biggl[\frac{1}{2s^5-1}\biggr]^n d(u,fu) \\ &\quad -\frac{1}{s^4}\sum_{i=0}^{n-1}\biggl[\frac{2s}{2s^5-1}\biggr]^{n-(i+1)}\varphi\biggl(\frac{d(f^{n-(i+1)}u,f^{n-(i)}u)+d(f^{n-(i)}u,f^{n-(i-1)}u)}{2s}\biggr) \\ &\leq d(u,fu) \\ &\quad -\frac{1}{s^4}\sum_{i=0}^{n-1}\biggl[\frac{2s}{2s^5-1}\biggr]^{n-(i+1)}\varphi\biggl(\frac{d(f^{n-(i+1)}u,f^{n-(i)}u)+d(f^{n-(i)}u,f^{n-(i-1)}u)}{2s}\biggr). \end{aligned}$$

Therefore,

$$\frac{1}{s^4}\sum_{i=0}^{n-1}\biggl[\frac{2s}{2s^5-1}\biggr]^{n-(i+1)}\varphi\biggl(\frac{d(f^{n-(i+1)}u,f^{n-(i)}u)+d(f^{n-(i)}u,f^{n-(i-1)}u)}{2s}\biggr)=0,$$

which from our assumptions as regards φ implies that

$$d\bigl(f^{n-(i+1)}u,f^{n-(i)}u\bigr)=d\bigl(f^{n-(i)}u,f^{n-(i-1)}u\bigr)=0$$

for all $0 \leq i \leq n-1$. Now, taking $i = n-1$, we have $u = fu$.

In the other case, the proof will be done in a similar way. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

1. Abbas, M, Dorić, D: Common fixed point theorem for four mappings satisfying generalized weak contractive condition. Filomat **24**(2), 1-10 (2010)
2. Abbas, M, Nazir, T, Radenović, S: Common fixed points of four maps in partially ordered metric spaces. Appl. Math. Lett. **24**, 1520-1526 (2011)
3. Abbas, M, Nazir, T, Radenović, S: Common fixed point of generalized weakly contractive maps in partially ordered *G*-metric spaces. Appl. Math. Comput. **218**, 9383-9395 (2012)
4. Abbas, M, Rhoades, BE: Common fixed point results for non-commuting mappings without continuity in generalized metric spaces. Appl. Math. Comput. **215**, 262-269 (2009)
5. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. J. Appl. Anal. **87**(1), 109-116 (2008)
6. Shatanawi, W, Postolache, M: Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces. Fixed Point Theory Appl. **2013**, 271 (2013)
7. Shatanawi, W, Postolache, M: Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces. Fixed Point Theory Appl. **2013**, 60 (2013)
8. Shatanawi, W, Postolache, M: Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. Fixed Point Theory Appl. **2013**, 54 (2013)
9. Choudhury, BS, Metiya, N, Postolache, M: A generalized weak contraction principle with applications to coupled coincidence point problems. Fixed Point Theory Appl. **2013**, 152 (2013)
10. Aydi, H, Karapınar, E, Postolache, M: Tripled coincidence point theorems for weak ϕ-contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, 44 (2012)
11. Aydi, H, Shatanawi, W, Postolache, M, Mustafa, Z, Tahat, N: Theorems for Boyd-Wong type contractions in ordered metric spaces. Abstr. Appl. Anal. **2012**, Article ID 359054 (2012)
12. Shatanawi, W, Pitea, A, Lazovic, R: Contraction conditions using comparison functions on *b*-metric spaces. Fixed Point Theory Appl. **2014**, 135 (2014)
13. Alber, YI, Guerre-Delabriere, S: Principle of weakly contractive maps in Hilbert spaces. In: Gohberg, I, Lyubich, Y (eds.) New Results in Operator Theory. Advances and Appl., vol. 98, pp. 7-22. Birkhäuser, Basel (1997)
14. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683-2693 (2001)
15. Zhang, Q, Song, Y: Fixed point theory for generalized φ-weak contractions. Appl. Math. Lett. **22**, 75-78 (2009)
16. Dorić, D: Common fixed point for generalized (ψ,φ)-weak contractions. Appl. Math. Lett. **22**, 1896-1900 (2009)
17. Moradi, S, Fathi, Z, Analouee, E: Common fixed point of single valued generalized φ_f-weak contractive mappings. Appl. Math. Lett. **24**(5), 771-776 (2011)
18. Razani, A, Parvaneh, V, Abbas, M: A common fixed point for generalized $(\psi,\varphi)_{f,g}$-weak contractions. Ukr. Math. J. **63**(11), 1756-1769 (2012)
19. Aghajani, A, Radenović, S, Roshan, JR: Common fixed point results for four mappings satisfying almost generalized (S,T)-contractive condition in partially ordered metric spaces. Appl. Math. Comput. **218**, 5665-5670 (2012)
20. Esmaily, J, Vaezpour, SM, Rhoades, BE: Coincidence point theorem for generalized weakly contractions in ordered metric spaces. Appl. Math. Comput. **219**, 1536-1548 (2012)
21. Harjani, J, Sadarangani, K: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72**(3-4), 1188-1197 (2010)
22. Nashine, HK, Samet, B: Fixed point results for mappings satisfying (ψ,φ)-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal. **74**, 2201-2209 (2011)
23. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
24. Nieto, JJ, Pouso, RL, Rodríguez-López, R: Fixed point theorems in ordered abstract sets. Proc. Am. Math. Soc. **135**, 2505-2517 (2007)
25. Shatanawi, W, Samet, B: On (ψ,ϕ)-weakly contractive condition in partially ordered metric spaces. Comput. Math. Appl. **62**, 3204-3214 (2011)
26. Nieto, JJ, Rodríguez-López, R: Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. Engl. Ser. **23**, 2205-2212 (2007)
27. Radenović, S, Kadelburg, Z: Generalized weak contractions in partially ordered metric spaces. Comput. Math. Appl. **60**, 1776-1783 (2010)
28. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some application to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
29. Czerwik, S: Nonlinear set-valued contraction mappings in *b*-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**(2), 263-276 (1998)
30. Akkouchi, M: Common fixed point theorems for two selfmappings of a *b*-metric space under an implicit relation. Hacet. J. Math. Stat. **40**(6), 805-810 (2011)
31. Aydi, H, Bota, M, Karapınar, E, Mitrović, S: A fixed point theorem for set-valued quasi-contractions in *b*-metric spaces. Fixed Point Theory Appl. **2012**, 88 (2012)
32. Boriceanu, M: Fixed point theory for multivalued generalized contraction on a set with two *b*-metrics. Stud. Univ. Babeş-Bolyai, Math. **LIV**(3), 3-14 (2009)

33. Boriceanu, M: Strict fixed point theorems for multivalued operators in b-metric spaces. Int. J. Mod. Math. **4**(3), 285-301 (2009)
34. Boriceanu, M, Bota, M, Petrusel, A: Multivalued fractals in b-metric spaces. Cent. Eur. J. Math. **8**(2), 367-377 (2010)
35. Bota, M, Molnar, A, Varga, C: On Ekeland's variational principle in b-metric spaces. Fixed Point Theory **12**(2), 21-28 (2011)
36. Hussain, N, Dorić, D, Kadelburg, Z, Radenović, S: Suzuki-type fixed point results in metric type spaces. Fixed Point Theory Appl. **2012**, 126 (2012)
37. Hussain, N, Shah, MH: KKM mappings in cone b-metric spaces. Comput. Math. Appl. **62**, 1677-1684 (2011)
38. Olatinwo, MO: Some results on multi-valued weakly Jungck mappings in b-metric space. Cent. Eur. J. Math. **6**(4), 610-621 (2008)
39. Pacurar, M: Sequences of almost contractions and fixed points in b-metric spaces. An. Univ. Vest. Timiş., Ser. Mat.-Inform. **XLVIII**(3), 125-137 (2010)
40. Singh, SL, Prasad, B: Some coincidence theorems and stability of iterative proceders. Comput. Math. Appl. **55**, 2512-2520 (2008)
41. Ansari, AH, Chandok, S, Ionescu, C: Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions. J. Inequal. Appl. **2014**, 429 (2014)
42. Altun, I, Simsek, H: Some fixed point theorems on ordered metric spaces and application. Fixed Point Theory Appl. **2010**, Article ID 621492 (2010)
43. Jungck, G: Compatible mappings and common fixed points. Int. J. Math. Math. Sci. **9**, 771-779 (1986)
44. Jungck, G: Common fixed points for noncontinuous nonself maps on nonmetric spaces. Far East J. Math. Sci. **4**, 199-215 (1996)
45. Radenović, S, Kadelburg, Z, Jandrlić, D, Jandrlić, A: Some results on weakly contractive maps. Bull. Iran. Math. Soc. **38**(3), 625-645 (2012)
46. Jovanović, M, Kadelburg, Z, Radenović, S: Common fixed point results in metric-type spaces. Fixed Point Theory Appl. **2010**, Article ID 978121 (2010)
47. Aghajani, A, Abbas, M, Roshan, JR: Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slovaca **64**(4), 941-960 (2014)
48. Khamsi, MA, Hussain, N: KKM mappings in metric type spaces. Nonlinear Anal. **73**(9), 3123-3129 (2010)
49. Khamsi, MA: Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. **2010**, Article ID 315398 (2010)
50. Chugh, R, Kadian, T, Rani, A, Rhoades, BE: Property P in G-metric spaces. Fixed Point Theory Appl. **2010**, Article ID 401684 (2010)
51. Jeong, GS, Rhoades, BE: Maps for which $F(T) = F(T^n)$. Fixed Point Theory Appl. **6**, 87-131 (2005)
52. Khandaqji, M, Al-Sharif, S, Al-Khaleel, M: Property P and some fixed point results on (ψ,φ)-weakly contractive G-metric spaces. Int. J. Math. Math. Sci. **2012**, Article ID 675094 (2012)