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POSITIVE SOLUTION OF A CERTAIN CLASS OF OPERATOR EQUATIONS*

ДОДАТНІ РОЗВ'ЯЗКИ ДЕЯКОГО КЛАСУ ОПЕРАТОРНИХ РІВНЯНЬ

The positive solutions of certain class of matrix equations have been recently considered by Bhatia et al. [Bull. London Math. Soc. – 2000. – **32**. – P. 214 – 228], [SIAM J. Matrix Anal. and Appl. – 1993. – **14**. – P. 132 – 136; 2005. – **27**. – P. 103 – 114], Kwong [Linear Algebra and Appl. – 1988. – **108**. – P. 177 – 197] and Cvetković and Milovanović [Linear Algebra and Appl. – 2008. – **429**. – P. 2401 – 2414]. Following the idea used in the last paper, we study a class of operator equations in infinite-dimensional spaces for which we prove that the positivity of a solution can be established provided that a certain rational function is positive semidefinite.

Додатні розв'язки деякого класу матричних рівнянь було нещодавно вивчено в роботах Бхатіа та ін. [Bull. London Math. Soc. – 2000. – **32**. – P. 214 – 228], [SIAM J. Matrix Anal. and Appl. – 1993. – **14**. – P. 132 – 136; 2005. – **27**. – P. 103 – 114], Квонга [Linear Algebra and Appl. – 1988. – **108**. – P. 177 – 197] та Цветковича та Міловановича [Linear Algebra and Appl. – 2008. – **429**. – P. 2401 – 2414]. З використанням ідеї, запропонованої в останній роботі, вивчено клас операторних рівнянь в нескінченновимірних просторах, для якого доведено, що додатність розв'язку можна встановити за умови, що деяка раціональна функція є позитивно напіввизначеною.

1. Introduction and preliminaries. Matrix equations appear in several fields of mathematics, e.g., linear algebra, differential equations, numerical analysis, optimization theory, etc. (cf. [1, 9, 11]). Also, these equations play important roles in many applications in system theory, e.g., stability analysis and optimal control (cf. [10, 20]), observer design [8], as well as in other computational sciences and engineering.

In a survey paper, Lancaster [18] reviewed the existence and uniqueness results, as well as the methods for obtaining explicit representations for a solution X for matrix equations of the form

$$\sum_{k=1}^{p} A_k X C_k = B, \tag{1.1}$$

with A_k, C_k, and B being known matrices not necessarily square. A special case of (1.1)

$$AX + XC = B \tag{1.2}$$

is known as the *Sylvester equation* (cf. [10], Chapter 9). A further important special case is obtained by putting $C = A^*$, where A^* is the conjugate transpose of A (or $C = A^T$ in the real case). Such an equation

$$AX + XA^* = B \tag{1.3}$$

is the well-known *Lyapunov equation*, which has been studied extensively. The equation (1.3) has a great deal with the analysis of the stability of motion (cf. [10, 20]).

In a recent paper [3], Bhatia and Drisi have considered the following matrix equations:

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$$\begin{aligned} AX + XA &= B, \\ A^2X + XA^2 + tAXA &= B, \\ A^3X + XA^3 + t(A^2XA + AXA^2) &= B, \\ A^4X + XA^4 + t(A^3XA + AXA^3) + 6A^2XA^2 &= B, \\ A^4X + XA^4 + 4(A^3XA + AXA^3) + tA^2XA^2 &= B, \end{aligned} \tag{1.4}$$

where A is a given positive definite matrix and matrix B is positive semidefinite. The first equation in (1.4) has the form (1.2), with $C = A$. The second equation has been studied by Kwong in [17], where he gave proof of the existence of the positive semidefinite solution. In [3] (see also [2, 4]) necessary and sufficient conditions for the parameter t were given in order that the previous equations have positive semidefinite solutions, provided that B is positive semidefinite matrix. There is also a strong connection between the question of positive semidefinite solutions of these equations and various inequalities involving unitarily equivalent matrix norms (see [2, 12, 13, 15, 16]).

Cvetković and Milovanović [7] have considered the existence of positive semidefinite solutions of a general matrix equation of the following form:

$$\sum_{\nu=0}^{m} a_\nu A^{m-\nu} X A^\nu = B, \tag{1.5}$$

where A is a positive definite matrix, B is a positive semidefinite matrix, $a_\nu = a_{m-\nu} \in \mathbb{R}$, $\nu = 0, 1, \ldots, m$, and $a_0 = a_m > 0$.

For problems connected with differential equations where it is necessary to consider the operator equations similar to those of the form (1.1), Daleckii and Krein in [9] (§ 3) considered general equations of the form

$$\sum_{j,k=0}^{n} c_{j,k} A^j X B^k = Y,$$

where B is a bounded linear operator on a certain Banach space B_1, A is a bounded linear operator on a certain Banach space B_2, and operator Y as well as the unknown operator X are bounded linear operators from space B_1 to B_2. They gave the conditions under which there exists a *unique solution* of such an operator equation (see [9], Theorem 3.2).

In this paper we continue with ideas presented in [7] and study a certain class of operator equations on infinite dimensional spaces.

Let V be a separable Hilbert space (for example ℓ^2) and let $\mathbf{B}(V)$ be the space of bounded linear operators on V. We consider the following operator equation:

$$\sum_{k=0}^{p} a_k A^k X A^{p-k} = B, \quad a_0, a_1, \ldots, a_n \in \mathbb{R}, \quad A, B \in \mathbf{B}(V), \tag{1.6}$$

with symmetry $a_k = a_{p-k}$, $k = 0, 1, \ldots, p$, and $a_0 = a_p > 0$. Our aim is to give conditions which ensure that there exists a *unique positive solution* of the equation (1.6).

To express results easier, we introduce the polynomial

$$q^p(x, y) = \sum_{k=0}^{p} a_k x^k y^{p-k}.$$

We note that q^p is a homogeneous polynomial of degree p, for which we require that the coefficients a_k, $k = 0, 1, \ldots, p$, be such that $q^p(x, y) > 0$, $x, y > 0$.

We want to use an information from the spectrum of the operator A to solve this equation. We assume that A and B are symmetric, and that A is strictly positive $((Ax, x) > 0, x \neq 0)$ and compact operator. In this settings B has to be compact, if we are looking for the continuous solution X, otherwise it need not. We assume that spectral resolutions of the linear operators A and B, provided B is compact, are given by

$$A = \sum_{k \in \mathbb{N}} \lambda_k^A P_k^A, \quad B = \sum_{k \in \mathbb{N}} \lambda_k^B P_k^B,$$

where P_k^A and P_k^B are orthogonal projections onto the eigenspaces corresponding to the eigenvalue λ_k^A of the operator A and to the eigenvalue λ_k^B of the operator B, respectively.

For the brevity we introduce the notation $q_{k,\ell}^p = q^p(\lambda_k^A, \lambda_\ell^A)$.

Also, we denote the identity operator simply by 1, which will not lead to confusion since it will be clear from the context when 1 denotes the identity operator and when it denotes the number.

Solution of the equation (1.6) need not be compact, even when both of operators A and B are compact. For example,

$$\sum_{k=0}^{p} A^k X A^{p-k} = (p+1)A^p$$

has the solution $X = 1$ which is not compact.

The paper is organized as follows. In Section 2 we present some auxiliary results. The main results on the positive solution of the operator equation (1.6), as well as two examples are given in Section 3.

2. Auxiliary results. The following result is well-known, but we present the proof for the sake of completeness.

Lemma 2.1. *Let P_k, $k \in \mathbb{N}$, be orthogonal projections with the properties*

$$P_k P_\ell = 0, \quad k \neq \ell, \quad k, \ell \in \mathbb{N} \quad \text{and} \quad \left(\sum_k P_k \right) x \to x, \quad x \in V.$$

If X is continuous, then for every $x \in V$ and every $\varepsilon > 0$ there exists n_0, such that for every $n, m > n_0$

$$\left\| \left(X - \sum_{k=1}^{m} \sum_{\ell=1}^{n} P_\ell X P_k \right) x \right\| < \varepsilon.$$

In other words, $\sum_{k,\ell} P_\ell X P_k$ converges to X strongly.

Proof. We obtain result easily. If $X = 0$ the statement is trivial. So, we assume that $X \neq 0$. First, due to property $P_k P_\ell = 0$, $k \neq \ell$ $k, \ell \in \mathbb{N}$, we know that $\sum_{k=1}^{n} P_k$ is an orthogonal projection. Fix $\varepsilon > 0$, then there exists $n_1 \in \mathbb{N}$ such that for $n > n_1$ we have

$$\left\| \left(\sum_{k=1}^{n} P_k \right) x - x \right\| < \frac{\varepsilon}{6\|X\|},$$

and there exists $n_2 \in \mathbb{N}$ such that for $n > n_2$ we obtain

$$\left\| \left(\sum_{k=1}^{n} P_k \right) Xx - Xx \right\| < \frac{\varepsilon}{2}.$$

Let $n_0 = \max\{n_1, n_2\}$. Then for $n, m > n_0$ we get

$$\left\| Xx - \sum_{k=1}^{m} \sum_{\ell=1}^{n} P_k X P_\ell x \right\| = \left\| Xx - \sum_{\ell=1}^{n} X P_\ell x + \sum_{\ell=1}^{n} X P_\ell x - \sum_{k=1}^{m} \sum_{\ell=1}^{n} P_k X P_\ell x \right\| \le$$

$$\le \left\| X \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| +$$

$$+ \left\| \left(1 - \sum_{k=1}^{m} P_k \right) X \sum_{\ell=1}^{n} P_\ell x - \left(1 - \sum_{k=1}^{m} P_k \right) Xx + \left(1 - \sum_{k=1}^{m} P_k \right) Xx \right\| \le$$

$$\le \|X\| \left\| \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| + \left\| \left(1 - \sum_{k=1}^{m} P_k \right) X \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| + \left\| \left(1 - \sum_{k=1}^{m} P_k \right) Xx \right\| \le$$

$$\le \|X\| \left\| \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| + \left(\|1\| + \left\| \sum_{k=1}^{m} P_k \right\| \right) \|X\| \left\| \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| +$$

$$+ \left\| \left(1 - \sum_{k=1}^{m} P_k \right) Xx \right\| \le$$

$$\le 3\|X\| \left\| \left(1 - \sum_{\ell=1}^{n} P_\ell \right) x \right\| + \left\| \left(1 - \sum_{k=1}^{m} P_k \right) Xx \right\| < \varepsilon.$$

This clearly proves the statement. We used the fact that $\left\| \sum_{k=1}^{n} P_k \right\| = 1$, since it is orthogonal projection.

Lemma 2.2. *Assume X is the continuous solution of* (1.6), *then*

$$P_\ell^A X P_k^A = (1/q_{\ell,k}^p) P_\ell^A B P_k^A, \qquad k, \ell \in \mathbb{N}.$$

If $B = 0$, then $X = 0$ is the unique continuous solution of (1.6). *If* (1.6) *has a continuous solution, then it is unique solution in the set of continuous solutions.*

Proof. Since $P_k^A A = A P_k^A = \lambda_k^A P_k^A$, $k \in \mathbb{N}$, we have

$$P_\ell^A B P_k^A = P_\ell^A \sum_{\nu=0}^{p} a_\nu A^\nu X A^{p-\nu} P_k^A = \sum_{\nu=0}^{p} a_\nu (\lambda_\ell^A)^\nu (\lambda_k^A)^{p-\nu} P_\ell^A X P_k^A =$$

$$= q^p(\lambda_\ell^A, \lambda_k^A) P_\ell^A X P_k^A.$$

Conclusion holds, since the operator A is strictly positive.

It is obvious that $X = 0$ is a solution of the homogeneous equation

$$\sum_{k=0}^{p} a_k A^k X A^{p-k} = 0.$$

But, for any $k, \ell \in \mathbb{N}$, we get $P_\ell^A 0 P_k^A = 0$, hence $P_\ell^A X P_k^A = 0$, for any continuous solution X. For any $x \in V$, we obtain $Xx = \sum_{k,\ell\in\mathbb{N}} 0x = 0$, hence, $X = 0$ is the unique solution.

If X_1 and X_2 are two continuous solutions of (1.6), then $X_1 - X_2$ is the solution of the homogeneous equation. Hence, $X_1 - X_2 = 0$ which proves our statement.

Theorem 2.1. *Let B be compact. The series*

$$\sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A$$

converges strongly if and only if equation (1.6) *has continuous solution.*

Proof. According to [19, p. 166], if the sequence of linear operators converges, then it strongly converges to a bounded linear operator. Hence, strongly convergent series $\sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A$ converges to some continuous X. To prove that X is a solution of (1.6), we note

$$\sum_{\nu=0}^{p} a_\nu A^\nu X A^{p-\nu} = \sum_{\nu=0}^{p} a_\nu A^\nu \left(\sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{k,\ell}^p} P_\ell^A B P_k^A \right) A^{p-\nu} =$$

$$= \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{k,\ell}^p} \sum_{\nu=0}^{p} a_\nu A^\nu P_\ell^A B P_k^A A^{p-\nu} =$$

$$= \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{k,\ell}^p} \left(\sum_{\nu=0}^{p} a_\nu (\lambda_\ell^A)^\nu (\lambda_\ell^A)^{p-\nu} \right) P_\ell^A B P_k^A = \sum_{k,\ell\in\mathbb{N}} P_\ell^A B P_k^A = B.$$

If X is a continuous solution of (1.6), then the series

$$\sum_{k,\ell\in\mathbb{N}} P_\ell^A X P_k^A = \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A,$$

converges strongly to X.

Example **2.1.** Let us illustrate the previous discussion using an example. Consider the case $P_k^A = P_k^B = (\cdot, e_k)e_k$, $k \in \mathbb{N}$, $p = 1$, $q_1(x, y) = x + y$ and $\lambda_k^A = (\lambda_k^B)^2/2 = 1/(2k^2)$, $k \in \mathbb{N}$, where $\{e_k\}_{k\in\mathbb{N}}$ is the Hilbert basis of V. Then

$$\frac{1}{\lambda_\ell^A + \lambda_k^A} P_\ell^A B P_k^A = \frac{\delta_{k,\ell}}{\lambda_k^B} P_k^B = \delta_{\ell,k} k P_k^B, \quad \ell, k \in \mathbb{N}. \tag{2.1}$$

Consequently, for $x = \sum_{m\in\mathbb{N}} (1/m) e_m$, we have

$$\frac{1}{\lambda_\ell^A + \lambda_k^A} P_\ell^A B P_k^A x = \delta_{\ell,k} k(x, e_k) e_k = \delta_{\ell,k} e_k.$$

Hence, the series

$$\sum_{k=1}^{n} \sum_{\ell=1}^{m} \frac{1}{\lambda_k^A + \lambda_\ell^A} P_\ell^A B P_k^A x = \sum_{k=1}^{\min\{m,n\}} e_k$$

is not convergent. This means that even if B is strictly positive and compact, a continuous solution need not exist (Theorem 2.1).

There is a special case in which we can claim that a continuous solution exists.

Theorem 2.2. *Let $B = CA^p$, $C \in \mathbf{B}(V)$, and $AC = CA$. Then the equation* (1.6) *has the solution $X = 1/q^p(1,1)C$.*

Proof. For $x \in V$, we have

$$\sum_{k=1}^{n} \sum_{\ell=1}^{m} \frac{1}{q_{k,\ell}^p} P_k^A C A^p P_\ell^A x = C \sum_{k=1}^{n} \sum_{\ell=1}^{m} \frac{(\lambda_k^A)^p}{q^p(\lambda_k^A, \lambda_\ell^A)} P_k^A P_\ell^A x =$$

$$= C \sum_{k=1}^{\min\{m,n\}} \frac{(\lambda_k^A)^p}{q^p(\lambda_k^A, \lambda_k^A)} P_k^A x = \frac{1}{q^p(1,1)} C \sum_{k=1}^{\min\{m,n\}} P_k^A x \to \frac{1}{q^p(1,1)} Cx,$$

where we use the fact that $AC = CA$ implies $P^A C = C P^A$, where P^A is any spectral projection of A (see [5, p. 150]).

In the sequel we are dealing with the unbounded solutions of (1.6). We denote

$$V_0^{A,n} = \bigoplus_{k=1}^{n} P_k^A V, \quad n \in \mathbb{N}, \quad V_0^A = \bigcup_{n \in \mathbb{N}} V_0^{A,n}.$$

Note that V_0^A is a linear space. Every $x \in V_0^A$ is a linear combination of eigenvectors of the operator A. In what follows we assume that B is a bounded operator, not necessarily compact. As a consequence, we are searching for unbounded solutions of (1.6).

Lemma 2.3. *Suppose $x \in V_0^A$, then the series $\sum_{k,\ell \in \mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A x$ converges.*

Proof. Since $x \in V_0^A$, by definition of V_0^A, there exists some $h \in \mathbb{N}$ such that $x \in V_0^{A,h}$. Hence, $\left(\sum_{k=1}^{h} P_k^A\right) x = x$ and $P_k^A x = 0$, $k > h$. For $m > h$ we have

$$\sum_{\ell=1}^{n} \sum_{k=1}^{m} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A x = \sum_{k=1}^{h} \left(\sum_{\ell=1}^{n} \frac{1}{q_{\ell,k}^p} P_\ell^A \right) (B P_k^A x) \to \sum_{k=1}^{h} (q^p(\lambda_k^A, A))^{-1} B P_k^A x,$$

as $n, m \to +\infty$, where we used the fact that $(q^p(\lambda_k^A, \cdot))^{-1} : \sigma(A) \to \mathbb{R}$, $k = 1, \ldots, h$, is bounded on the spectrum of A.

Lemma 2.4. *The closure of V_0^A coincides with V.*

Proof. Take any $x \in V$, then $V_0^A \ni \sum_{k=1}^{n} P_k^A x \to x$, as $n \to +\infty$.

Lemma 2.5. *Let $X_0 : V_0^A \to V$ be the operator defined by*

$$X_0 x = \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A x.$$

Then X_0 is a linear operator densely defined on V. Operator X_0 is symmetric and closable.

For any $x \in V_0^A$, we have

$$\sum_{\nu=0}^{p} a_\nu A^\nu X_0 A^{p-\nu} x = Bx.$$

Proof. According to Lemma 2.3, we see that X_0 is well defined. If $x, y \in V_0^A$ and if α, β are scalars, then $\alpha x + \beta y \in V_0^A$, and according to

$$X_0(\alpha x + \beta y) = \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A (\alpha x + \beta y) =$$

$$= \alpha \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A x + \beta \sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{\ell,k}^p} P_\ell^A B P_k^A y = \alpha X_0 x + \beta X_0 y,$$

we infer that X_0 is a linear operator. Hence, according to Lemma 2.4, X_0 is a densely defined linear operator on V.

We prove that X_0 is symmetric. Let $x, y \in V_0^A$, then

$$(X_0 x, y) = \lim_{n\to+\infty} \left(\sum_{k,\ell=1}^{n} \frac{1}{q_{k,\ell}^p} P_k^A B P_\ell^A x, y \right) =$$

$$= \lim_{n\to+\infty} \left(x, \sum_{k,\ell=1}^{n} \frac{1}{q_{k,\ell}^p} P_k^A B P_\ell^A y \right) = (x, X_0 y).$$

We prove that X_0 is closable. According to [5, p. 66], we have to prove that if $x_k \in D(X_0)$, $\lim x_k = 0$ and $\lim X_0 x_k = y$, then $y = 0$.

Let $z \in V_0^A$ be arbitrary. Then we get

$$(y, z) = \lim_{n\to+\infty} (X_0 x_n, z) = \lim_{n\to+\infty} \left(\lim_{m\to+\infty} \sum_{k,\ell=1}^{m} \frac{1}{q_{k,\ell}^p} P_k^A B P_\ell^A x_n, z \right) =$$

$$= \lim_{n\to+\infty} \left(x_n, \lim_{m\to+\infty} \sum_{k,\ell=1}^{m} \frac{1}{q_{k,\ell}^p} P_k^A B P_\ell^A z \right) =$$

$$= \lim_{n\to+\infty} (x_n, X_0 z) = (0, X_0 z) = 0.$$

Since V_0^A is dense in V, we get $y = 0$. Thus, we conclude that X_0 is closable.

Choose now an arbitrary $x \in V_0^{A,n}$. Then $A^\nu x \in V_0^{A,n}$, $\nu = 0, 1, \ldots, p$. We conclude that the left-hand side of (1.6) is well defined and we have

$$\sum_{\nu=0}^{p} a_\nu A^\nu X_0 A^{p-\nu} x = \sum_{\nu=0}^{p} a_\nu A^\nu \left(\sum_{k,\ell\in\mathbb{N}} \frac{1}{q_{k,\ell}^p} P_\ell^A B P_k^A \right) A^{p-\nu} x =$$

$$= \sum_{k,\ell\in\mathbb{N}} P_\ell^A B P_k^A x = Bx.$$

Definition 2.1. *Operator X is the minimal closed extension of X_0. We call X the solution of the equation* (1.6).

Trivially X is symmetric, as being closure of a symmetric operator X_0. We call X the solution, despite the fact, that we can claim that equation (1.6) is valid only on V_0^A.

There is a special case in which we can give some stronger results.

Lemma 2.6. *Let $B(V_0^A) \subset V_0^A$. The solution X is self-adjoint.*

Proof. If B satisfies the mentioned condition, then clearly solution of the equation (1.6), on the set V_0^A, can be given by

$$Xx = \sum_{k,\ell=1}^{n} \frac{1}{q_{k,\ell}^p} P_k^A B P_\ell^A x,$$

where $n = \max\{m_1, m_2\}$ and m_1 is such that $x \in V_0^{A,m_1}$ and m_2 is such that $BP_\ell^A x \in V_0^{A,m_2}$, $\ell = 1, \ldots, m_1$. Since X is closure of X_0, we know that $X \restriction_{V_0^A} = X_0$ and $(X-\lambda) \restriction_{V_0^A} = X_0 - \lambda$, where $A \restriction_{V_0^A}$ denotes restriction of operator A to V_0^A.

Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ be arbitrary and choose $x \in \ker(X_0 - \lambda)$. Then, we have $\lambda(x,x) = (X_0 x, x) = (x, X_0 x) = \overline{\lambda}(x,x)$. It follows that $x = 0$ and $\ker(X_0 - \lambda) = \{0\}$ for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Let us fix $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $x \in \operatorname{rang}(X_0 - \lambda)^\perp \cap V_0^A$. For $y \in V_0^A$ arbitrary, we have $0 = ((X_0 - \lambda)y, x) = (y, (X_0 - \overline{\lambda})x)$. We conclude $x \in \ker(X_0 - \overline{\lambda}) = \{0\}$. Hence, it must be $\operatorname{rang}(X_0 - \lambda) = V_0^A$ for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Let X^* denote the adjoint of X. Since X is densely defined and closed, we know that there exists X^*, which is closed and densely defined. Also (see [5, p. 70])

$$\overline{\operatorname{rang}(X-\lambda)} \bigoplus \ker(X^* - \overline{\lambda}) = V.$$

Since X is symmetric on $D(X)$, we get $X \subset X^*$ (see [5, p. 97]). For $\lambda \in \mathbb{C}\backslash\mathbb{R}$, we conclude that

$$\ker(X^* - \lambda) = (\operatorname{rang}(X - \overline{\lambda}))^\perp \subset (\operatorname{rang}(X_0 - \overline{\lambda}))^\perp = (V_0^A)^\perp = \{0\}.$$

According to von Neumann's formulae (see [5, p. 106]), we know that

$$D(X^*) = D(X) \dot{+} \ker(X^* - \lambda) \dot{+} \ker(X^* - \overline{\lambda}),$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is arbitrary. We conclude directly that $D(X^*) = D(X)$, which gives $X = X^*$ and X is self-adjoint.

According to the previous lemmas we are ready to formulate the following statement.

Theorem 2.3. *Let A and B be symmetric, and let A be strictly positive and compact and B bounded. There exists symmetric and closed X such that the equation* (1.6) *is valid on V_0^A. Moreover, if $B(V_0^A) \subset V_0^A$, then X is self-adjoint.*

It is interesting to give an interpretation of the example given in (2.1). It can be easily seen that $X = \sum_{k\in\mathbb{N}} k(\cdot, e_k)e_k$, is actually, spectral resolution of the self-adjoint operator X.

Another obvious interpretation of the result is for the case $B = 2$. Then, X is the solution of the equation $AX + XA = 2$. Due to symmetry of A and $\ker(A) = \{0\}$, we know that A does not have residual spectrum. Consequently, the range of A has to be dense in V and there must exist the self-adjoint inverse of A.

3. Positive solutions. We denote by $2\mathbb{N}$ and $2\mathbb{N}-1$ the sets of even and odd positive integers. Consider now the linear operators $I_k : P_k^A V \to \mathbb{C}^{\operatorname{rang}(P_k^A)}$, $k \in \mathbb{N}$, defined on some orthonormal basis $H_k = \{e_{k,1}, \ldots, e_{k,\operatorname{rang}(P_k^A)}\}$, $k \in \mathbb{N}$, by $I_k e_{k,\ell} = f_\ell$, $\ell = 1, \ldots, \operatorname{rang}(P_k^A)$, $k \in \mathbb{N}$, where $\{f_1, \ldots, f_{\operatorname{rang}(P_k^A)}\}$ is the natural basis of $\mathbb{C}^{\operatorname{rang}(P_k^A)}$, and respective direct sum $I_0^n = \oplus_{k=1}^n I_k$. It is trivial fact that $I_0^n : V_0^{A,n} \to \mathbb{C}^{\dim(V_0^{A,n})}$ is an isometrical isomorphism.

In what follows we adopt the following definitions:

An operator A is nonnegative, positive, strictly positive if and only if for all $x \in D(A)$ we have $(Ax, x) \geq 0$, $(Ax, x) \geq 0$ and $A \neq 0$, $(Ax, x) > 0$, respectively.

A matrix A is positive definite, positive semidefinite if and only if $(Ax, x) > 0$, $(Ax, x) \geq 0$, respectively.

A function $f : \mathbb{R} \to \mathbb{C}$ is positive definite if and only if for all $n \in \mathbb{N}$ and any given points x_k, $k = 1, \ldots, n$, the matrix $\|f(x_k - x_\ell)\|_{k,\ell=1}^n$, is positive semidefinite.

Lemma 3.1. *If the function $x \mapsto \varphi_p(x)$, given by*

$$\frac{1}{\varphi_p(x)} = \begin{cases} \displaystyle\sum_{\nu=0}^{p/2-1} a_\nu \cosh\left(\frac{p}{2} - \nu\right) x + \frac{1}{2} a_{p/2}, & p \in 2\mathbb{N}, \\ \displaystyle\sum_{\nu=0}^{(p-1)/2} a_\nu \cosh\left(\frac{p}{2} - \nu\right) x, & p \in 2\mathbb{N} - 1, \end{cases}$$

is positive definite, then the linear operator $C_n : V_0^{A,n} \to V_0^{A,n}$, defined by

$$C_n x = \sum_{k,\ell=1}^n \frac{1}{q_{\ell,k}^p} (P_k^A x, (I_0^n)^{-1} 1_v) P_\ell^A (I_0^n)^{-1} 1_v, \tag{3.1}$$

where $1_v = (1, 1, 1, \ldots, 1) \in \mathbb{C}^{\dim(V_0^{A,n})}$, is positive and the matrix $I_0^n C_n (I_0^n)^{-1}$ is positive semidefinite.

Proof. It is easy to prove that C_n is a symmetric linear operator. For $x \in V_0^{A,n}$, we have

$$(C_n x, x) = \left(\sum_{k,\ell=1}^n \frac{1}{q_{\ell,k}^p} (P_k^A x, (I_0^n)^{-1} 1_v) P_\ell^A (I_0^n)^{-1} 1_v, x \right) =$$

$$= \sum_{k,\ell=1}^n \frac{1}{q_{\ell,k}^p} (P_k^A x, (I_0^n)^{-1} 1_v)(P_\ell^A (I_0^n)^{-1} 1_v, x) =$$

$$= \sum_{k,\ell=1}^n \frac{1}{q_{\ell,k}^p} (P_k^A x, (I_0^n)^{-1} 1_v)((I_0^n)^{-1} 1_v, P_\ell^A x) =$$

$$= \sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^{p}} (P_k^A x, (I_0^n)^{-1} 1_v) \overline{(P_\ell^A x, (I_0^n)^{-1} 1_v)}.$$

Accordingly, we note that if matrix $D_n = \|1/q_{\ell,k}^p\|_{\ell,k=1}^n$ is positive semidefinite, the operator C_n is positive. To prove semidefiniteness of D_n we use the same arguments as in [7]. Since λ_k, $k \in \mathbb{N}$, is a positive sequence, we can represent it as $\lambda_k = e^{x_k}$, $x_k \in \mathbb{R}$, $k \in \mathbb{N}$. Then we conclude that

$$\left\| \frac{1}{q^p(\lambda_\ell^A, \lambda_k^A)} \right\| = \operatorname{diag}(e^{px_\ell/2}) \left\| \frac{1}{\sum_{\nu=0}^{p} a_\nu e^{(p/2-\nu)(x_\ell - x_k)}} \right\| \operatorname{diag}(e^{px_k/2}) =$$

$$= Z \|\varphi_p(x_\ell - x_k)\| Z^*.$$

We recognize that the matrix D_n is congruent with the matrix

$$E_n = ||\varphi_p(x_\ell - x_k)||,$$

hence, positive semidefiniteness of D_n and E_n are equivalent. The matrix Z is simply diagonal matrix with the positive entries $1/\sqrt{2}e^{px_k/2}$, $k \in \mathbb{N}$. According to the condition of this lemma, the matrix E_n is positive semidefinite, hence, matrix D_n is positive semidefinite, and C_n is positive.

Positive semidefiniteness of the matrix $I_0^n C_n (I_0^n)^{-1}$ is a consequence of a positivity of the operator C_n, since, for $x \in \mathbb{C}^{\dim(V_0^{A,n})}$, we get

$$(I_0^n C_n (I_0^n)^{-1} x, x) = (C_n (I_0^n)^{-1} x, (I_0^n)^{-1} x) = (C_n((I_0^n)^{-1} x), (I_0^n)^{-1} x).$$

Lemma 3.2. *Let B be positive operator and*

$$B_n = \sum_{\ell=1}^{n} \sum_{k=1}^{m} P_\ell^A B P_k^A, \quad n \in \mathbb{N}.$$

Then $\{B_n\}$ is a sequence of the nonnegative linear operators, $B = \lim B_n$, and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operator B_n is positive.

Proof. According to Lemma 2.1 and continuity of B, a sequence of the linear operators

$$B_n = \sum_{\ell=1}^{n} \sum_{k=1}^{n} P_\ell^A B P_k^A = \left(\sum_{\ell=1}^{n} P_\ell^A \right) B \left(\sum_{k=1}^{n} P_k^A \right), \quad n \in \mathbb{N},$$

converges to B. Even more, we see that B_n, $n \in \mathbb{N}$, is the sequence of nonnegative operators, since for every $x \in V_0^{A,n}$, we have

$$(B_n x, x) = \left(\left(\sum_{\ell=1}^{n} P_\ell^A \right) B \left(\sum_{k=1}^{n} P_k^A \right) x, x \right) = \left(B \left(\sum_{k=1}^{n} P_k^A \right) x, \left(\sum_{k=1}^{n} P_k^A \right) x \right) \geq 0.$$

Since $0 \neq B = \lim B_n$ it follows that there exists $n_0 \in \mathbb{N}$ such that $B_n \neq 0$ for all $n \geq n_0$.

Theorem 3.1. *Let us assume that operator B is positive and that the function $x \mapsto \varphi_p(x)$, given by*

$$\frac{1}{\varphi_p(x)} = \begin{cases} \sum\limits_{\nu=0}^{p/2-1} a_\nu \cosh\left(\frac{p}{2}-\nu\right) x + \frac{1}{2} a_{p/2}, & p \in 2\mathbb{N}, \\ \sum\limits_{\nu=0}^{(p-1)/2} a_\nu \cosh\left(\frac{p}{2}-\nu\right) x, & p \in 2\mathbb{N}-1, \end{cases}$$

is positive definite. The solution of the operator equation (1.6), *described in Theorem* 2.3, *is positive.*

Proof. Let X be a solution of (1.6). Then we have strong convergence of the sequence $X_n = \sum_{k=1}^{n} \sum_{\ell=1}^{n} P_\ell^A X P_k^A$, as $n \to +\infty$, on V_0^A.

Let C_n be an operator defined in (3.1). The linear operators $I_0^n B_n (I_0^n)^{-1}$, $I_0^n C_n (I_0^n)^{-1}$ and $I_0^n X_n (I_0^n)^{-1}$ on $\mathbb{C}^{\dim V_0^{A,n}}$, can be represented using matrix multiplication as matrices. Even more, the matrix $I_0^n X_n (I_0^n)^{-1}$ is a Schur product of the matrices $I_0^n C_n (I_0^n)^{-1}$ and $I_0^n B_n (I_0^n)^{-1}$. However, for $n \geq n_0$ the matrix $I_0^n B_n (I_0^n)^{-1}$ is positive semidefinite, due to positivity of B_n, and

$$(I_0^n B_n (I_0^n)^{-1} a, a) = (B_n (I_0^n)^{-1} a, (I_0^n)^{-1} a) \geq 0, \quad a \in \mathbb{C}^{\dim(V_0^{A,n})}.$$

The matrix $I_0^n C_n (I_0^n)^{-1}$ is positive semidefinite according to Lemma 3.1. Accordingly, for all $n \geq n_0$ the operator X_n is positive, since

$$0 \leq (I_0^n X_n (I_0^n)^{-1} a, a) = (X_n (I_0^n)^{-1} a, (I_0^n)^{-1} a), \quad a \in \mathbb{C}^{\dim(V_0^{A,n})}.$$

Using this observations, we simply derive that for every $x \in V_0^A$ we have

$$(Xx, x) = \lim_{n\to+\infty} (X_n x, x) = \lim_{n\to+\infty} \sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^p} (P_\ell X P_k x, x) =$$

$$= \lim_{n\to+\infty} \sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^p} (X P_k x, P_\ell x) = \lim_{n\to+\infty} \sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^p} (X P_k \pi_n x, P_\ell \pi_n x) =$$

$$= \lim_{n\to+\infty} \sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^p} (P_\ell X P_k \pi_n x, \pi_n x) =$$

$$= \lim_{n\to+\infty} \left(\sum_{k,\ell=1}^{n} \frac{1}{q_{\ell,k}^p} P_\ell X P_k \pi_n x, \pi_n x \right) = \lim_{n\to+\infty} (X_n \pi_n x, \pi_n x) \geq 0,$$

and $X \neq 0$, where we used the fact that $P_k \pi_n = P_k$, $k = 1, \ldots, n$, for $\pi_n = \sum_{k=1}^{n} P_k$, which is orthogonal projection onto $V_0^{A,n}$.

For every $x \in D(X)$ there exists a sequence $x_n \in V_0^A$, such that $x_n \to x$, $X x_n \to Xx$ as $n \to +\infty$. Therefore, since $(X x_n, x_n) \geq 0$, we have $(Xx, x) = \lim (X x_n, x_n) \geq 0$.

As in [7] we can define a characteristic polynomial for the equation (1.6).

Definition 3.1. *For even p we define the characteristic polynomial Q^p for the equation* (1.6) *to be $Q^p(\cosh t) = 1/\varphi_p(t)$, and for odd p we define the corresponding characteristic polynomial to be $Q^p(\cosh t) = 1/(\cosh(t/2)\varphi_p(t))$.*

Now, we can use this characteristic polynomial to give the following statement.

Theorem 3.2. *Suppose we are given the equation* (1.6), *with a strictly positive and compact operator A, with the characteristic polynomial Q^p which has k_1 real zeros contained in the interval $[-1, 1)$ and k_2 zeros smaller than -1, with $k_1 \geq k_2$, for even p, and $k_1 + 1 \geq k_2$, for odd p, where $k_1 + k_2 = [p/2]$. Then the corresponding function φ_p is positive definite, i.e., the equation* (1.6) *has a positive symmetric and closed solution, provided B is positive.*

Proof. It is proved in [7] that under this condition function φ_p is positive definite. According to Theorem 3.1, in this case we have a symmetric and closed solution.

We give now an example with integral operators acting on the space $L^2(0, 1)$. Denote by $C^2[0, 1]$ the space of twice continuously-differentiable functions on $[0, 1]$ and by $H^2[0, 1]$ the corresponding space of twice differentiable functions with the second derivative being an element of $L^2(0, 1)$. In addition, let $C_0^2[0, 1]$ and $H_0^2[0, 1]$ be their subspaces, with the additional conditions

$$f'(0) + f'(1) = 0 \quad \text{and} \quad f'(1) = f(0) + f(1), \tag{3.2}$$

respectively.

We need also $C^4[0, 1]$ as the space of four times continuously-differentiable functions on $[0, 1]$ and $H^4[0, 1]$ as the space of four times differentiable functions with fourth derivative being an element of $L^2(0, 1)$. With $C_0^4[0, 1]$ and $H_0^2[0, 1]$ we denote their subspaces, with the additional conditions (3.2) and

$$f'''(0) + f'''(1) = 0 \quad \text{and} \quad f'''(1) = f''(0) + f''(1), \tag{3.3}$$

respectively.

Lemma 3.3. *Let an integral operator $C : L^2(0, 1) \to L^2(0, 1)$ be defined by*

$$(Cf)(x) = \int_0^1 |x - t| f(t)\, dt, \quad x \in [0, 1].$$

Then $\ker(C) = \{0\}$, $\operatorname{rang}(C) = H_0^2[0, 1]$, $\overline{\operatorname{rang}(C)} = L^2(0, 1)$, *$C$ is compact and self-adjoint and $D^2C = CD^2 = 2$, where $D^2 : C_0^2[0, 1] \to L^2(0, 1)$ is the second derivative and $\overline{C_0^2[0, 1]} = L^2(0, 1)$.*

Operator $A = C^2$ is strictly positive, self-adjoint, with $\ker(A) = \{0\}$, $\operatorname{rang}(A) = H_0^4[0, 1]$, $\overline{\operatorname{rang}(A)} = L^2(0, 1)$ *and $D^4A = AD^4 = 4$, where $D^4 : C_0^4[0, 1] \to L^2(0, 1)$ is the forth derivative and $\overline{C_0^4[0, 1]} = L^2(0, 1)$.*

Proof. For every $x \in [0, 1]$ we have $|x - t| \in L^2(0, 1)$, so that C is defined everywhere on $L^2(0, 1)$. We know that C is compact and self-adjoint since its kernel $|x - t|$ is continuous and symmetric (see [14, 21]). Let ε_n be a sequence of real numbers converging to zero. For a fixed $x \in [0, 1]$, consider the sequence of functions

$$g_n(t) = \frac{|x + \varepsilon_n - t| - |x - t|}{\varepsilon_n}, \quad n \in \mathbb{N}.$$

We have an integrable and uniform bound

$$\left|\frac{|x+\varepsilon_n-t|-|x-t|}{\varepsilon_n}\right| \leq \frac{|x+\varepsilon_n-t-x+t|}{|\varepsilon_n|}=1\in L^2(0,1),$$

as well as the pointwise convergence

$$\lim_{n\to+\infty} g_n(t)=\begin{cases}1, & x>t,\\ -1, & x<t,\end{cases}=g(t)\in L^2(0,1).$$

Using the Lebesgue theorem on dominated convergence (see [6]), we get

$$\lim_{n\to+\infty}\int_0^1 g_n(t)f(t)\,dt=\int_0^1 \lim_{n\to+\infty} g_n(t)f(t)\,dt=$$

$$=\int_0^1 \operatorname{sgn}(x-t)f(t)\,dt=(Cf)'(x).$$

Let ε_n, $n\in\mathbb{N}$, be again a sequence of real numbers converging to zero. Then

$$\lim_{n\to+\infty}\frac{1}{\varepsilon_n}\int_0^1 \left(\operatorname{sgn}(x+\varepsilon_n-t)-\operatorname{sgn}(x-t)\right)f(t)\,dt=$$

$$=2\lim_{n\to+\infty}\frac{1}{\varepsilon_n}\int_{[x,x+\varepsilon_n]} f(t)\,dt=2f(x),$$

for a.e. $x\in[0,1]$, according to the Lebesgue differentiation theorem (see [6]). Since

$$(Cf)'(0)=-\int_0^1 f(t)\,dt,\quad (Cf)'(1)=\int_0^1 f(t)\,dt,$$

$$(Cf)(0)+(Cf)(1)=\int_0^1 f(t)\,dt,$$

we see that Cf satisfies the conditions (3.2). Hence, for every $f\in L^2(0,1)$ we have $Cf\in H_0^2[0,1]$ and $2f(x)=(Cf)''(x)$, for a.e. $x\in[0,1]$. Therefore, $D^2C=2$.

On the other hand, using an integration by parts, for $f\in C_0^2[0,1]$ we get

$$(CD^2f)(x)=\int_0^1 |x-t|f''(t)\,dt=2f(x)+x(f'(0)-f'(1))+f'(1)-(f(0)+f(1)),$$

and, due to the conditions (3.2), we find $CD^2=2$.

If $f\in\ker(C)$, we have $f(x)=1/2(Cf)''(x)=1/2(0)''(x)=0$, for a.e. $x\in[0,1]$. We conclude that $\ker(C)=\{0\}$. Finally, it is a trivial fact that $\overline{C_0^2[0,1]}=\overline{H_0^2[0,1]}=L^2(0,1)$, where the closure is taken in L^2-norm.

The statement for the operator A can be obtained in the same fashion.

Lemma 3.4. *The eigenvalues of the operator* A *given in Lemma* 3.3 *are given by* $\lambda_k = \widetilde{\lambda}_k^2$, $k \in \mathbb{N}_0$, *where* $\widetilde{\lambda}_0 = 2/\alpha_0^2$, $\widetilde{\lambda}_k = -2/\nu_k^2$, $k \in \mathbb{N}$, α_0 *is the unique solution of the equation* $4 + e^{-\alpha}(2+\alpha) + e^{\alpha}(2-\alpha) = 0$ *and* ν_k, $k \in \mathbb{N}$, *are positive solutions of the equation* $2 + 2\cos\nu + \nu\sin\nu = 0$. *The corresponding eigenvectors are*

$$f_0(x) = \frac{1+e^{-\alpha_0}}{1+e^{\alpha_0}} e^{\alpha_0 x} + e^{-\alpha_0 x}, \quad f_k(x) = \frac{1+\cos\nu_k}{\sin\nu_k}\cos\nu_k x + \sin\nu_k x, \quad k \in \mathbb{N}.$$

Proof. We first find eigenvalues of the operator C. Starting with the equation $(Cf)(x) = \lambda f(x)$, by differentiating we get the following differential equation:

$$\lambda f''(x) - 2f(x) = 0, \tag{3.4}$$

with the boundary conditions (3.2). It is obvious that $\lambda = 0$ is not an eigenvalue.

For $\lambda > 0$, the solution of the differential equation (3.4) is given by $f(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$, $C_1, C_2 \in \mathbb{R}$, where $\alpha = \sqrt{2/\lambda}$. Using the boundary conditions (3.2), for C_1 and C_2, we get the system of linear equations

$$C_1 - C_2 + C_1 e^{\alpha} - C_2 e^{-\alpha} = 0,$$

$$C_1 + C_2 + C_1 e^{\alpha} + C_2 e^{-\alpha} = \alpha C_1 e^{\alpha} - \alpha C_2 e^{-\alpha},$$

which determinant is given by

$$\Delta = 4\cosh\frac{\alpha}{2}\left(2\cosh\frac{\alpha}{2} - \alpha\sinh\frac{\alpha}{2}\right).$$

The equation $\Delta = 0$ has the unique solution $\alpha = \alpha_0 > 0$. Thus, the operator C has one eigenvalue $\widetilde{\lambda}_0 = 2/\alpha_0^2$ greater than zero, and the corresponding eigenvector is

$$f_0(x) = \frac{1+e^{-\alpha_0}}{1+e^{\alpha_0}} e^{\alpha_0 x} + e^{-\alpha_0 x}.$$

For $\lambda < 0$, the solution of differential equation (3.4) is given by $f(x) = C_1\cos\nu x + C_2\sin\nu x$, $C_1, C_2 \in \mathbb{R}$, where $\nu = \sqrt{-2/\lambda}$. Using the boundary conditions (3.2) we get the following system of linear equations for C_1, C_2:

$$C_2 - C_1\sin\nu + C_2\cos\nu = 0,$$

$$C_1 + C_1\cos\nu + C_2\sin\nu = -\nu C_1\sin\nu + \nu C_2\cos\nu.$$

Therefore, $C_1 = C_2(1+\cos\nu)/\sin\nu$ and, since $C_2 \neq 0$ (because we are looking for nontrivial solutions), we get $2\cos\nu/2(\cos\nu/2 + \nu\sin\nu/2) = 0$. It is easy to see that if $\cos\nu/2 = 0$, then $C_1 = C_2 = 0$, and the values for ν are not eigenvalues of the operator C. Let us denote by ν_k, $k \in \mathbb{N}$, the positive solutions of the previous equations (one solution in each of intervals of the form $[k\pi, (k+1)\pi)$, $k \in \mathbb{N}_0$). Then, $\widetilde{\lambda}_k = -2/\nu_k^2$, $k \in \mathbb{N}$, are eigenvalues of the operator C, and

$$f_k(x) = \frac{1+\cos\nu_k}{\sin\nu_k}\cos\nu_k x + \sin\nu_k x, \quad k \in \mathbb{N},$$

are the corresponding eigenvectors.

It is easy to see that the eigenvalues of the operator $A = C^2$ are given by $\lambda_k = \widetilde{\lambda}_k^2$, $k \in \mathbb{N}_0$, and that f_k, $k \in \mathbb{N}_0$, are the corresponding eigenvectors ($\lambda_k f_k = \widetilde{\lambda}_k C f_k = C(\widetilde{\lambda}_k f_k) = C^2 f_k$).

Example **3.1.** Consider the equation

$$\sum_{k=0}^{p} A^k X_p A^{p-k} = 1, \quad p = 1, \tag{3.5}$$

where A is an operator given in Lemma 3.3. Since $V_0^A = 1(V_0^A)$, according to Lemma 2.6, the solution X_1 of the equation (3.5) is self-adjoint.

Let us denote by $\{e_k\}_{k\in\mathbb{N}_0}$ the orthonormal set of eigenvectors of the operator A. Then $A = \sum_{k=0}^{\infty} \lambda_k P_{\lambda_k}$, where P_{λ_k}, $k \in \mathbb{N}_0$, is a projection onto the eigenspace which correspond to the eigenvector λ_k. Since $P_{\lambda_k} = (\cdot, e_k)e_k$, $k \in \mathbb{N}_0$, we get

$$Af = \sum_{k=0}^{\infty} \lambda_k (f, e_k)e_k, \quad f \in L^2(0,1).$$

The solution of the equation (3.5) can be easily found as $X_1 = 1/8D^4$.

For $p = 1$ we get $Q^1(t) = 1$, and the solution X_1 is positive, according to Theorem 3.2. For $f \in C_0^4[0,1]$, a direct computation gives

$$(D^4 f, f) = \int_0^1 (D^4 f)(t) f(t)\, dt =$$

$$= (D^3 f)(t) f(t)\Big|_0^1 - (D^2 f)(t)(Df)(t)\Big|_0^1 + \int_0^1 ((D^2 f)(t))^2\, dt.$$

Using the boundary conditions (3.2) and (3.3) we get

$$(D^3 f)(1) f(1) - (D^3 f)(0) f(0) - (D^2 f)(1)(Df)(1) + (D^2 f)(0)(Df)(0) =$$

$$= (D^3 f)(1)((Df)(1) - f(0)) - (D^3 f)(0) f(0) -$$

$$-(D^2 f)(1)(Df)(1) - (D^2 f)(0)(Df)(1) =$$

$$= (D^3 f)(1)(Df)(1) - f(0)((D^3 f)(1) + (D^3 f)(0)) -$$

$$-(Df)(1)((D^2 f)(1) + (D^2 f)(0)) = 0.$$

Therefore,

$$(D^4 f, f) = \int_0^1 ((D^2 f)(t))^2\, dt \geq 0.$$

Example **3.2.** Consider the equation

$$\sum_{k=0}^{p} A^k X_p A^{p-k} = 1, \quad p = 2, \tag{3.6}$$

where A is an operator given in Lemma 3.3. In the same way as in Example 3.1, we conclude that the solution $X_2 = 1/48D^8$ of the equation (3.6) is self-adjoint. Since for $p = 2$ we have $Q^2(t) = t+1/2$, the solution X_2 is positive, according to Theorem 3.2.

Similarly as in Example 3.1, for $f \in C_0^8[0,1] = \{f \in C^8[0,1] \mid f^{(5)}(1) = f^{(4)}(0) + f^{(4)}(1), f^{(5)}(0) + f^{(5)}(1) = 0, f^{(7)}(1) = f^{(6)}(0) + f^{(6)}(1), f^{(7)}(0) + f^{(7)}(1) = 0\}$ we get

$$(D^8 f, f) = \int_0^1 (D^8 f)(t) f(t)\, dt =$$

$$= (D^7 f)(t) f(t)\Big|_0^1 - (D^6 f)(t)(Df)(t)\Big|_0^1 + \int_0^1 ((D^4 f)(t))^2\, dt = \int_0^1 ((D^4 f)(t))^2\, dt \geq 0.$$

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