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Quadrature Rules With an Even Number of Multiple Nodes and a Maximal Trigonometric Degree of Exactness

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Abstract. This paper is devoted to the interpolatory quadrature rules with an even number of multiple nodes, which have the maximal trigonometric degree of exactness. For constructing of such quadrature rules we introduce and consider the so–called $s-$ and $\sigma-$orthogonal trigonometric polynomials. We present a numerical method for construction of mentioned quadrature rules. Some numerical examples are also included.

1. Introduction

For a nonnegative integer n, by $\mathcal{T}_n$ we denote the linear space of all trigonometric polynomials of degree less than or equal to n, by $\widetilde{\mathcal{T}}_n$ the linear space $\mathcal{T}_n \ominus \operatorname{span}\{\sin nx\}$ or $\mathcal{T}_n \ominus \operatorname{span}\{\cos nx\}$, and by $\mathcal{T}$ the set of all trigonometric polynomials.

Let us suppose that a function w is integrable and nonnegative on the interval $[-\pi, \pi)$, vanishing there only on a set of a measure zero. We consider a quadrature rule of the following type

$$\int_{-\pi}^{\pi} f(x)\, w(x)\, \mathrm{d}x = \sum_{\nu=1}^{2n} \sum_{j=0}^{2s_\nu} A_{j,\nu} f^{(j)}(x_\nu) + R_n(f), \tag{1}$$

where s_ν, $\nu = 1, 2, \ldots, 2n$, are nonnegative integers.

Definition 1.1. *The quadrature rule of the form* (1) *has trigonometric degree of exactness equal to d if $R_d(f) = 0$ for every $f \in \mathcal{T}_d$, and if there exists $g \in \mathcal{T}_{d+1}$ for which $R_d(g) \neq 0$.*

We are interested in quadrature rule of the form (1) which has the maximal trigonometric degree of exactness. We consider the questions of existence and uniqueness of such quadrature rules, as well as their numerical construction. Let us notice that the quadrature rule (1) has an even number of nodes. The corresponding quadrature rules with an odd number of nodes were first considered in [8] for constant

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weight function $w(x) = 1$ and the case when all of the nodes have the same multiplicity. Quadrature rules with fixed number of free nodes of fixed different multiplicities at nodes, but again only for the constant weight function $w(x) = 1$, were considered in [4]. Finally, the general case of quadrature rules with an odd number of nodes with different multiplicities and with respect to arbitrary weight function, were considered in [9].

In this paper we pay our attention to quadrature rules with an even number of nodes and the maximal trigonometric degree of exactness. We finish this Section with some known results for a generalized Gaussian problem, given in [8]. Section 2 is devoted to quadrature rules with simple nodes. The case when all of the nodes have the same multiplicity is considered in Section 3, while the case of different multiplicities at nodes is considered in Section 4. Finally, the numerical method for constructing of considered quadrature rules is given in Section 5, where one numerical example is given, too.

1.1. A generalized Gaussian problem

Let the quadrature formula

$$\int_a^b f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{m}\sum_{j=0}^{N-1} A_{j,\nu} f^{(j)}(x_\nu) + R(f) \tag{2}$$

be such that $E(f) = 0 \Rightarrow R(f) = 0$, where E is a linear differential operator of order N. According to [8, p. 28] quadrature rules which are exact for all trigonometric polynomials of degree less than or equal to ν are relative to the differential operator (of order $2\nu + 1$): $E = \dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\prod_{k=1}^{\nu}\left(\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right)$.

Ghizzeti and Osicini in [8] considered whether there can exist a rule of the form

$$\int_a^b f(x)w(x)\,\mathrm{d}x = \sum_{\nu=1}^{m}\sum_{j=0}^{N-p_\nu-1} A_{j,\nu} f^{(j)}(x_\nu) + R(f), \tag{3}$$

with fixed integers p_ν, $0 \le p_\nu \le N-1$, $\nu = 1,2,\dots,m$, such that at least one of the integers p_ν is greater than or equal to 1, satisfying $E(f) = 0 \Rightarrow R(f) = 0$, too. They proved the following theorem (see [8, p. 45]).

Theorem 1.2. *For the given nodes $x_1, x_2, \dots, x_m$, the linear differential operator E of order N and the nonnegative integers $p_1, p_2, \dots, p_m$, $0 \le p_\nu \le N-1$, $\nu = 1,2,\dots,m$ $((\exists \nu \in \{1,2,\dots,m\})\ p_\nu \ge 1)$, consider the following homogenous boundary differential problem*

$$E(f) = 0, \quad f^{(j)}(x_\nu) = 0, \quad j = 0,1,\dots,N-p_\nu-1, \quad \nu = 1,2,\dots,m. \tag{4}$$

If this problem has no non–trivial solutions (whence $N \le mN - \sum_{\nu=1}^{m} p_\nu$) it is possible to write a quadrature rule of the type (3) *with $mN - \sum_{\nu=1}^{m} p_\nu - N$ parameters chosen arbitrarily. If, on the other hand, the problem* (4) *has q linearly independent solutions U_r, $r = 1,2,\dots,q$, with $N - mN + \sum_{\nu=1}^{m} p_\nu \le q \le p_\nu$ for all $\nu = 1,2,\dots,m$, then the formula* (3) *may apply only if the following q conditions*

$$\int_a^b U_r(x)\,w(x)\,\mathrm{d}x = 0, \quad r = 1,2,\dots,q, \tag{5}$$

are satisfied; if so, $mN - \sum_{\nu=1}^{m} p_\nu - N + q$ parameters in the formula (3) *can be chosen arbitrary.*

2. Quadrature rules with simple nodes

First we consider the special case when $s_1 = s_2 = \cdots = s_{2n} = 0$, i.e., the quadrature rule with simple nodes

$$\int_{-\pi}^{\pi} f(x)\,w(x)\,\mathrm{d}x = \sum_{\nu=1}^{2n} w_\nu f(x_\nu) + R_n(f). \tag{6}$$

For a given set of different nodes x_ν ($\in [-\pi,\pi)$), $\nu = 1,2,\ldots,2n$, quadrature rule (6) of the interpolatory type is exact for every $t \in \mathcal{T}_{n-1}$, and in addition for $t \in \widetilde{\mathcal{T}_n}$, but not for all $t \in \mathcal{T}_n$ (see [2]). If the nodes $x_\nu(\in [-\pi,\pi))$, $\nu = 1,2,\ldots,2n$, are not specified in advanced, one can try to find them such that quadrature rule (6) has the maximal trigonometric degree of exactness, i.e., such that quadrature rule (6) is exact for every $t \in \mathcal{T}_{2n-1}$.

Lemma 2.1. *The trigonometric polynomial of degree n,*

$$T_n(x) = c_0 + \sum_{k=1}^{n} (c_k \cos kx + d_k \sin kx), \quad c_k \in \mathbb{R}\ (k = 0,1,\ldots,n), \quad d_k \in \mathbb{R}\ (k = 1,\ldots,n), \quad c_n^2 + d_n^2 \neq 0, \tag{7}$$

which is orthogonal on $[-\pi,\pi)$ with respect to the weight function w to every trigonometric polynomial from $\mathcal{T}_{n-1}$ is determined uniquely, if the values of its leading coefficients, c_n and d_n, are given.

Proof. The orthogonality conditions can be written in the form

$$\int_{-\pi}^{\pi} T_n(x) \cos \nu x\, w(x)\, \mathrm{d}x = 0, \quad \nu = 0,1,\ldots,n-1,$$
$$\int_{-\pi}^{\pi} T_n(x) \sin \nu x\, w(x)\, \mathrm{d}x = 0, \quad \nu = 1,2,\ldots,n-1.$$

By changing $T_n(x)$ in these equations and introducing the following notations

$$\int_{-\pi}^{\pi} \cos \nu x \cos kx\, w(x)\, \mathrm{d}x = \alpha_{k,\nu}, \quad k,\nu = 0,1,\ldots,n,$$
$$\int_{-\pi}^{\pi} \cos \nu x \sin kx\, w(x)\, \mathrm{d}x = \beta_{k,\nu}, \quad \nu = 0,1,\ldots,n,\ k = 1,2,\ldots,n,$$
$$\int_{-\pi}^{\pi} \sin \nu x \sin kx\, w(x)\, \mathrm{d}x = \gamma_{k,\nu}, \quad k,\nu = 1,2,\ldots,n,$$

we obtain the following system of linear equations for determining the unknown coefficients $c_0, c_1, d_1, \ldots, c_{n-1}, d_{n-1}$:

$$c_0\alpha_{0,\nu} + \sum_{k=1}^{n-1} (c_k\alpha_{k,\nu} + d_k\beta_{k,\nu}) = -c_n\alpha_{n,\nu} - d_n\beta_{n,\nu}, \quad \nu = 0,1,\ldots,n-1, \tag{8}$$
$$c_0\beta_{\nu,0} + \sum_{k=1}^{n-1} (c_k\beta_{\nu,k} + d_k\gamma_{\nu,k}) = -c_n\beta_{\nu,n} - d_n\gamma_{\nu,n}, \quad \nu = 1,2,\ldots,n-1.$$

The determinant of this system is equal to

$$\Delta = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \beta_{1,0} & \alpha_{2,0} & \beta_{2,0} & \cdots & \alpha_{n-1,0} & \beta_{n-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \beta_{1,1} & \alpha_{2,1} & \beta_{2,1} & \cdots & \alpha_{n-1,1} & \beta_{n-1,1} \\ \beta_{1,0} & \beta_{1,1} & \gamma_{1,1} & \beta_{1,2} & \gamma_{1,2} & \cdots & \beta_{1,n-1} & \gamma_{1,n-1} \\ \alpha_{0,2} & \alpha_{1,2} & \beta_{1,2} & \alpha_{2,2} & \beta_{2,2} & \cdots & \alpha_{n-1,2} & \beta_{n-1,2} \\ \beta_{2,0} & \beta_{2,1} & \gamma_{2,1} & \beta_{2,2} & \gamma_{2,2} & \cdots & \beta_{2,n-1} & \gamma_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0,n-1} & \alpha_{1,n-1} & \beta_{1,n-1} & \alpha_{2,n-1} & \beta_{2,n-1} & \cdots & \alpha_{n-1,n-1} & \beta_{n-1,n-1} \\ \beta_{n-1,0} & \beta_{n-1,1} & \gamma_{n-1,1} & \beta_{n-1,2} & \gamma_{n-1,2} & \cdots & \beta_{n-1,n-1} & \gamma_{n-1,n-1} \end{vmatrix}.$$

Since $\alpha_{k,\nu} = \alpha_{\nu,k}$, $k,\nu = 0,1,\ldots,n-1$, and $\gamma_{k,\nu} = \gamma_{\nu,k}$, $k,\nu = 1,2,\ldots,n-1$, the determinant Δ is symmetric.

Now, we consider the following quadratic form in variables $\xi_0, \xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_{n-1}, \eta_{n-1}$:

$$\begin{aligned} F &= \sum_{\nu=0}^{n-1} \sum_{k=0}^{n-1} \alpha_{k,\nu} \xi_k \xi_\nu + 2 \sum_{\nu=0}^{n-1} \sum_{k=1}^{n-1} \beta_{k,\nu} \xi_\nu \eta_k + \sum_{\nu=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{k,\nu} \eta_k \eta_\nu \\ &= \int_{-\pi}^{\pi} w(x) \left[\sum_{\nu=0}^{n-1} \xi_\nu \cos \nu x \right] \cdot \left[\sum_{k=0}^{n-1} \xi_k \cos kx \right] \mathrm{d}x + 2 \int_{-\pi}^{\pi} w(x) \left[\sum_{\nu=0}^{n-1} \xi_\nu \cos \nu x \right] \cdot \left[\sum_{k=1}^{n-1} \eta_k \sin kx \right] \mathrm{d}x \\ &\quad + \int_{-\pi}^{\pi} w(x) \left[\sum_{\nu=1}^{n-1} \eta_\nu \sin \nu x \right] \cdot \left[\sum_{k=1}^{n-1} \eta_k \sin kx \right] \mathrm{d}x \\ &= \int_{-\pi}^{\pi} w(x) \left[\sum_{\nu=0}^{n-1} \xi_\nu \cos \nu x + \sum_{k=1}^{n-1} \eta_k \sin kx \right]^2 \mathrm{d}x > 0. \end{aligned}$$

The quadratic form F is positive and its determinant is equal to Δ, thus, $\Delta > 0$, and the system of linear equations (8) has the unique solution for the unknown coefficients $c_0, c_1, d_1, \ldots, c_{n-1}, d_{n-1}$. □

Obviously,

$$T_n(x) = A \prod_{\nu=1}^{2n} \sin \frac{x - x_\nu}{2}, \quad A \neq 0 \text{ is a constant}, \tag{9}$$

is a trigonometric polynomial of degree n. To the contrary, every trigonometric polynomial of degree n of the form (7) can be represented in the form (9) with

$$A = (-1)^n 2^{2n-1} \mathrm{i}(c_n - \mathrm{i}d_n) \mathrm{e}^{\mathrm{i}/2 \sum_{\nu=1}^{2n} x_\nu},$$

where $x_1, x_2, \ldots, x_{2n}$ are the zeros of the trigonometric polynomial (7), that lie in the strip $-\pi \leq \operatorname{Re} x < \pi$ (see [10]).

By using similar arguments as in [15], it is easy to prove the following result.

Lemma 2.2. *Every trigonometric polynomial of degree* $2n-1$,

$$T_{2n-1}(x) = a_0 + \sum_{k=1}^{2n-1} (a_k \cos kx + b_k \sin kx),$$

can be uniquely represented in the form $T_{2n-1}(x) = A_n(x) B_{n-1}(x) + \widetilde{R}_n(x)$, *where* $A_n(x)$ *is a certain trigonometric polynomial of degree n and* $B_{n-1}(x)$, $\widetilde{R}_n(x)$ *are wanted polynomials from* $\mathcal{T}_{n-1}$ *and* $\widetilde{\mathcal{T}}_n$, *respectively.*

By virtue of Lemma 2.2, simulating the development of the famous Gaussian quadrature rules for algebraic polynomials, the following result can be easily proved.

Theorem 2.3. *The interpolatory type quadrature rule* (6) *is of Gaussian type, i.e., it is exact for every* $t \in \mathcal{T}_{2n-1}$, *if and only if the nodes* x_ν, $\nu = 1, 2, \ldots, 2n$, *are the zeros of trigonometric polynomial* $T_n(x)$, *which is orthogonal on* $[-\pi, \pi)$ *with respect to the weight function* $w(x)$ *to every trigonometric polynomial from* $\mathcal{T}_{n-1}$.

It is well known that trigonometric polynomial of degree n can not have more than $2n$ distinct zeros in $[-\pi, \pi)$ (see [10]). Now we prove that the zeros of orthogonal trigonometric polynomials are all simple.

Theorem 2.4. *The trigonometric polynomial* $T_n \in \mathcal{T}_n$ *which is orthogonal on* $[-\pi, \pi)$ *with respect to the weight function* $w(x)$ *to every trigonometric polynomial from* $\mathcal{T}_{n-1}$, *has in* $[-\pi, \pi)$ *exactly* $2n$ *distinct simple zeros.*

Proof. The trigonometric polynomial T_n must have at least one zero of odd multiplicity in $[-\pi,\pi)$. Indeed, if we assume the contrary, then for $n \in \mathbb{N}$ we obtain that

$$\int_{-\pi}^{\pi} T_n(x)\cos 0x\, w(x)\,\mathrm{d}x = 0,$$

which is impossible, because the integrand does not change its sign on $[-\pi,\pi)$. Also, T_n must change its sign an even number of times (see [1, 2]).

Let us now suppose that the number of zeros of T_n on $[-\pi,\pi)$ is $2m$, $m < n$. We denote these zeros by $y_1, y_2, \dots, y_{2m}$, and set

$$t(x) = \prod_{k=1}^{2m} \sin\frac{x-y_k}{2}.$$

Since $t \in \mathcal{T}_m$, $m < n$, we have $\int_{-\pi}^{\pi} T_n(x)t(x)\,w(x)\,\mathrm{d}x = 0$, which again gives a contradiction, since the integrand does not change its sign on $[-\pi,\pi)$.

Therefore, T_n must have exactly $2n$ different simple zeros on $[-\pi,\pi)$. □

3. Quadrature rules with multiple nodes with the same multiplicity and *s*–orthogonal trigonometric polynomials

In this section we consider quadrature rule of the form (1) where $s_1 = s_2 = \cdots = s_{2n} = s > 0$, i.e.,

$$\int_{-\pi}^{\pi} f(x)\,w(x)\,\mathrm{d}x = \sum_{\nu=1}^{2n}\sum_{j=0}^{2s} A_{j,\nu} f^{(j)}(x_\nu) + R_n(f), \tag{10}$$

which has the maximal trigonometric degree of exactness, i.e., such that $R_n(f) = 0$ for all $f \in \mathcal{T}_{2n(s+1)-1}$. The boundary differential problem (4) in this case has the following form

$$E(f) = 0, \quad f^{(j)}(x_\nu) = 0, \quad j = 0,1,\dots,2s, \quad \nu = 1,2,\dots,2n, \tag{11}$$

where

$$E = \frac{\mathrm{d}}{\mathrm{d}x}\prod_{k=1}^{2n(s+1)-1}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right)$$

is the differential operator of order $N = 4n(s+1) - 1$.

For $n > 0$ the boundary problem (11) has $2n-1$ linear independent non–trivial solutions (see [8, p. 141]):

$$U_\ell(x) = \left(\prod_{\nu=1}^{2n} \sin\frac{x-x_\nu}{2}\right)^{2s+1} \cos\ell x, \quad \ell = 0,1,\dots,n-1,$$

$$V_\ell(x) = \left(\prod_{\nu=1}^{2n} \sin\frac{x-x_\nu}{2}\right)^{2s+1} \sin\ell x, \quad \ell = 1,2,\dots,n-1.$$

According to Theorem 1.2, for $s_1 = s_2 = \cdots = s_{2n} = s$, the nodes $x_1, x_2, \dots, x_{2n}$ of the quadrature rule (10) satisfy the following conditions

$$\int_{-\pi}^{\pi}\left(\prod_{\nu=1}^{2n} \sin\frac{x-x_\nu}{2}\right)^{2s+1} \cos\ell x\, w(x)\,\mathrm{d}x = 0, \quad \ell = 0,1,\dots,n-1,$$

$$\int_{-\pi}^{\pi}\left(\prod_{\nu=1}^{2n} \sin\frac{x-x_\nu}{2}\right)^{2s+1} \sin\ell x\, w(x)\,\mathrm{d}x = 0, \quad \ell = 1,2,\dots,n-1,$$

i.e., $x_1, x_2, \dots, x_{2n}$ are the zeros of the trigonometric polynomial $T_{s,n}$ which satisfies the following equation

$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+1} t_{n-1}(x)\, w(x)\, \mathrm{d}x = 0, \quad \text{for arbitrary } t_{n-1} \in \mathfrak{T}_{n-1}. \tag{12}$$

We call such trigonometric polynomials as *s–orthogonal trigonometric polynomials* with respect to the weight function $w(x)$ on $[-\pi, \pi)$.

In order to prove the existence and uniqueness of $T_{s,n}(x)$ we use the following well–known facts about the best approximation (see [3, p. 58–60]).

Remark 3.1. *Let X be a Banach space and Y be a closed linear subspace of X. For each $f \in X$, the error of approximation of f by elements from Y is defined as $\inf_{g\in Y} \|f - g\|$. If there exists some $g = g_0 \in Y$ for which that infimum is attained, then g_0 is called the best approximation to f from Y. For each finite dimensional subspace X_n of X and each $f \in X$, there exists the best approximation to f from X_n. In addition, if X is a strictly convex space, then each $f \in X$ has at most one element of the best approximation in each closed linear subspace $Y \subset X$.*

Theorem 3.2. *Trigonometric polynomial $T_{s,n}(x)$, with given leading coefficients, which is s–orthogonal on $[-\pi, \pi)$ with respect to a given weight function $w(x)$ is determined uniquely.*

Proof. Let us set $X = L^{2s+2}[-\pi, \pi]$, $u = w(x)^{1/(2s+2)}(c_n \cos nx + d_n \sin nx) \in L^{2s+2}[-\pi, \pi]$, and fix the following $2n - 1$ linearly independent elements in $L^{2s+2}[-\pi, \pi]$:

$$u_j = w(x)^{1/(2s+2)} \cos jx, \quad j = 0, 1, \dots, n-1, \quad v_k = w(x)^{1/(2s+2)} \sin kx, \quad k = 1, 2, \dots, n-1.$$

Here, $Y = \operatorname{span}\{u_0, u_1, v_1, u_2, v_2, \dots, u_{n-1}, v_{n-1}\}$ is a finite dimensional subspace of X and, according to Remark 3.1, for each element from X there exists the best approximation from Y, i.e., there exist $2n - 1$ constants α_j, $j = 0, 1, \dots, n-1$, β_k, $k = 1, 2, \dots, n-1$, such that the error

$$\left\| u - \Big(\alpha_0 u_0 + \sum_{k=1}^{n-1} (\alpha_k u_k + \beta_k v_k)\Big) \right\| = \left(\int_{-\pi}^{\pi} \Big(c_n \cos nx + d_n \sin nx - \Big(\alpha_0 + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx)\Big)\Big)^{2s+2} w(x)\, \mathrm{d}x \right)^{1/(2s+2)},$$

is minimal, i.e., for every n and for every choice of the leading coefficients c_n, d_n, $c_n^2 + d_n^2 \neq 0$, there exists a trigonometric polynomial of degree n

$$T_{s,n}(x) = c_n \cos nx + d_n \sin nx - \Big(\alpha_0 + \sum_{k=1}^{n-1} (\alpha_k \cos kx + \beta_k \sin kx)\Big),$$

such that

$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+2}\, w(x)\, \mathrm{d}x$$

is minimal. Since the space $L^{2s+2}[-\pi, \pi]$ is strictly convex, according to Remark 3.1, the problem of the best approximation has the unique solution, i.e., the trigonometric polynomial $T_{s,n}$ is unique.

It follows that for each of the following $2n - 1$ functions

$$F_k^C(\lambda) = \int_{-\pi}^{\pi} (T_{s,n}(x) + \lambda \cos kx)^{2s+2}\, w(x)\, \mathrm{d}x, \quad k = 0, 1, \dots, n-1,$$
$$F_k^S(\lambda) = \int_{-\pi}^{\pi} (T_{s,n}(x) + \lambda \sin kx)^{2s+2}\, w(x)\, \mathrm{d}x, \quad k = 1, 2, \dots, n-1,$$

its derivative must be equal to zero for $\lambda = 0$. Therefore, we get

$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+1} \cos kx\, w(x)\, \mathrm{d}x = 0, \quad k = 0, 1, \dots, n-1,$$
$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+1} \sin kx\, w(x)\, \mathrm{d}x = 0, \quad k = 1, 2, \dots, n-1,$$

which means that the polynomial $T_{s,n}(x)$ satisfies s−orthogonality conditions (12). □

Theorem 3.3. *Trigonometric polynomial $T_{s,n}(x)$ which is s–orthogonal on $[-\pi,\pi)$ with respect to a given weight function $w(x)$ has in $[-\pi,\pi)$ exactly $2n$ distinct simple zeros.*

Proof. The trigonometric polynomial $T_{s,n}(x)$ has on $[-\pi,\pi)$ at least one zero of odd multiplicity. If we assume the contrary, for $n \geq 1$ we obtain the following contradiction to (12)

$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+1} \cos 0x \, w(x) \, \mathrm{d}x \neq 0,$$

since $(T_{s,n}(x))^{2s+1}$ does not change its sign on $[-\pi,\pi)$. Also, $T_{s,n}$ must change its sign on $[-\pi,\pi)$ an even number of times (see [1, 2]).

Let us now suppose that the number of zeros of $T_{s,n}$ on $[-\pi,\pi)$ of odd multiplicities is $2m$, $m < n$. Let us denote these zeros by $y_1, y_2, \ldots, y_{2m}$ and set

$$t(x) = \prod_{k=1}^{2m} \sin \frac{x - y_k}{2}.$$

Since $t \in \mathcal{T}_m$, $m < n$, we get

$$\int_{-\pi}^{\pi} (T_{s,n}(x))^{2s+1} t(x) \, w(x) \, \mathrm{d}x = 0,$$

which is a contradiction, because the integrand does not change its sign on $[-\pi,\pi)$.

Therefore, $T_{s,n}$ must have exactly $2n$ different simple zeros on $[-\pi,\pi)$. □

4. Quadrature rules with multiple nodes with different multiplicities and σ–orthogonal trigonometric polynomials

Let us denote $\sigma = (s_1, s_2, \ldots, s_{2n})$ and $N_1 = \sum_{\nu=1}^{2n}(s_\nu + 1) - 1$. We study quadrature rules of the form (1), which have maximal trigonometric degree of exactness, i.e., for which $R_n(f) = 0$ for all $f \in \mathcal{T}_{N_1}$. In this case boundary differential problem (4) has the following form

$$E(f) = 0, \quad f^{(j)}(x_\nu) = 0, \quad j = 0, 1, \ldots, 2s_\nu, \quad \nu = 1, 2, \ldots, 2n, \tag{13}$$

where

$$E = \frac{\mathrm{d}}{\mathrm{d}x} \prod_{k=1}^{N_1} \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2 \right)$$

is a differential operator of order $N = 2N_1 + 1$.

The boundary problem (13) has the following $2n - 1$ linear independent nontrivial solutions

$$U_\ell(x) = \prod_{\nu=1}^{2n} \left(\sin \frac{x - x_\nu}{2} \right)^{2s_\nu+1} \cos \ell x, \quad \ell = 0, 1, \ldots, n-1,$$

$$V_\ell(x) = \prod_{\nu=1}^{2n} \left(\sin \frac{x - x_\nu}{2} \right)^{2s_\nu+1} \sin \ell x, \quad \ell = 1, 2, \ldots, n-1.$$

According to Theorem 1.2, the nodes $x_1, x_2, \ldots, x_{2n}$ of the quadrature rule (1) satisfy conditions

$$\int_{-\pi}^{\pi} \prod_{\nu=1}^{2n} \left(\sin \frac{x - x_\nu}{2} \right)^{2s_\nu+1} \cos \ell x \, w(x) \, \mathrm{d}x = 0, \quad \ell = 0, 1, \ldots, n-1,$$

$$\int_{-\pi}^{\pi} \prod_{\nu=1}^{2n} \left(\sin \frac{x - x_\nu}{2} \right)^{2s_\nu+1} \sin \ell x \, w(x) \, \mathrm{d}x = 0, \quad \ell = 1, 2, \ldots, n-1,$$

i.e.,

$$\int_{-\pi}^{\pi} \prod_{\nu=1}^{2n} \left(\sin \frac{x - x_\nu}{2}\right)^{2s_\nu+1} t(x)\, w(x)\, \mathrm{d}x = 0, \quad \text{for all } t \in \mathcal{T}_{n-1}. \tag{14}$$

Trigonometric polynomial

$$T_{\sigma,n}(x) = \prod_{\nu=1}^{2n} \sin \frac{x - x_\nu}{2}$$

which satisfies condition (14) will be called *σ–orthogonal trigonometric polynomial* with respect to the weight function $w(x)$ on $[-\pi, \pi)$.

Since the dimension of $\mathcal{T}_{n-1}$ is $2n-1$, we have $2n-1$ orthogonality conditions. The σ–orthogonal trigonometric polynomial of degree n has $2n+1$ coefficients, which means that two of them can be fixed in advanced. Alternatively, if we directly compute the nodes of the σ−orthogonal trigonometric polynomial of degree n, we can fix one of them in advance since for $2n$ nodes we have $2n-1$ orthogonality conditions. Using the notation from the Theorem 1.2, we have

$$mN - \sum_{\nu=1}^{m} p_\nu - N + q = 2nN - \sum_{\nu=1}^{2n} (N - 2s_\nu - 1) - 2\Big(\sum_{\nu=1}^{2n} (s_\nu + 1) - 1\Big) - 1 + 2n - 1 = 0.$$

Therefore, if one of the nodes is fixed, the quadrature rule (1) is unique.

By the same arguments as in Theorem 3.3 and using conditions (14) one can prove the following theorem.

Theorem 4.1. *The σ–orthogonal trigonometric polynomial $T_{\sigma,n}$ with respect to the weight function $w(x)$ on $[-\pi, \pi)$ has exactly $2n$ distinct simple zeros on $[-\pi, \pi)$.*

The existence of σ–orthogonal trigonometric polynomials will be proved using theory of implicitly defined orthogonality. The existence of implicity defined orthogonal algebraic polynomials was proved in [6], while the existence of implicitly defined orthogonal trigonometric polynomials of semi–integer degree was proved in [9]. Here we prove the existence of implicitly defined orthogonal trigonometric polynomials.

Theorem 4.2. *Let p be a nonnegative continuous function, vanishing only on a set of a measure zero. Then there exists a trigonometric polynomial T_n, of degree n, orthogonal on $[-\pi, \pi)$ to every trigonometric polynomial of degree less than or equal to $n-1$ with respect to the weight function $p(T_n(x))w(x)$.*

Proof. Let us denote by $\widehat{\mathcal{T}}_n$ the set of all trigonometric polynomials of degree n which have $2n$ real distinct zeros and $-\pi$ as one of the zeros, i.e., which have the zeros $-\pi = x_1 < x_2 < \cdots < x_{2n} < \pi$, and $S_{2n-1} = \{\mathbf{x} = (x_2, x_3, \ldots, x_{2n}) \in \mathbb{R}^{2n-1} : -\pi < x_2 < x_3 < \cdots < x_{2n} < \pi\}$. For a given function p and an arbitrary $Q_n \in \widehat{\mathcal{T}}_n$, we introduce the inner product as follows

$$\langle f, g\rangle_{Q_n} = \int_{-\pi}^{\pi} f(x) g(x)\, p(Q_n(x)) w(x)\, \mathrm{d}x, \quad f, g \in \mathcal{T}.$$

It is obvious that there is one to one correspondence between the sets $\widehat{\mathcal{T}}_n$ and S_{2n-1}, which means that for every element $\mathbf{x} = (x_2, x_3, \ldots, x_{2n}) \in S_{2n-1}$ and for

$$Q_n(x) = \cos \frac{x}{2} \prod_{\nu=2}^{2n} \sin \frac{x - x_\nu}{2} \tag{15}$$

there exists a unique system of orthogonal trigonometric polynomials $U_k \in \widehat{\mathcal{T}}_k$, $k = 0, 1, \ldots, n$, such that

$$\langle U_n, \cos kx\rangle_{Q_n} = \langle U_n, \sin jx\rangle_{Q_n} = 0, \quad k = 0, 1, \ldots, n-1, \;\; j = 1, 2, \ldots, n-1, \quad \langle U_n, U_n\rangle_{Q_n} \neq 0.$$

In such a way, we introduce a mapping $F_n : S_{2n-1} \to S_{2n-1}$, defined in the following way: for any $\mathbf{x} = (x_2, x_3, \ldots, x_{2n}) \in S_{2n-1}$ we have $F_n(\mathbf{x}) = \mathbf{y}$, where $\mathbf{y} = (y_2, y_3, \ldots, y_{2n}) \in S_{2n-1}$ is such that $-\pi, y_2, y_3, \ldots, y_{2n}$ are the zeros of the orthogonal trigonometric polynomial of degree n with respect to the weight function $p(Q_n(x))w(x)$, where $Q_n(x)$ is given by (15). For an arbitrary $\mathbf{x} = (x_2, x_3, \ldots, x_{2n}) \in \overline{S}_{2n-1} \backslash S_{2n-1}$, the function $p\Big(\cos(x/2) \prod_{\nu=2}^{2n} \sin((x - x_\nu)/2)\Big) w(x)$ is an admissible weight function, too.

We are going to prove that F_n is continuous mapping on $\overline{S}_{2n-1}$. Let $\mathbf{x} \in \overline{S}_{2n-1}$ be an arbitrary point, $\{\mathbf{x}^{(m)}\}$, $m \in \mathbb{N}$, a convergent sequence of points from S_{2n-1}, which converges to $\mathbf{x}$, $\mathbf{y} = F_n(\mathbf{x})$, and $\mathbf{y}^{(m)} = F_n(\mathbf{x}^{(m)})$, $m \in \mathbb{N}$. Let $\mathbf{y}^* \in \overline{S}_{2n-1}$ be an arbitrary limit point of the sequence $\{\mathbf{y}^{(m)}\}$ when $m \to \infty$. Thus,

$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - y_\nu^{(m)}}{2} \cos kx\, p\left(\cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - x_\nu^{(m)}}{2}\right) w(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n-1,$$
$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - y_\nu^{(m)}}{2} \sin jx\, p\left(\cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - x_\nu^{(m)}}{2}\right) w(x)\,\mathrm{d}x = 0, \quad j = 1, 2, \ldots, n-1.$$

According to Lebesgue Theorem of dominant convergence (see [13, p. 83]), when $m \to +\infty$ we obtain

$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - y_\nu^{*}}{2} \cos kx\, p\left(\cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - x_\nu}{2}\right) w(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n-1,$$
$$\int_{-\pi}^{\pi} \cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - y_\nu^{*}}{2} \sin jx\, p\left(\cos\frac{x}{2} \prod_{\nu=2}^{2n} \sin\frac{x - x_\nu}{2}\right) w(x)\,\mathrm{d}x = 0, \quad j = 1, 2, \ldots, n-1.$$

Hence, $\cos(x/2) \prod_{\nu=2}^{2n} \sin((x - y_\nu^*)/2)$ is the trigonometric polynomial of degree n which is orthogonal to all trigonometric polynomials of degree less than or equal to $n-1$ with respect to the weight function $p\Big(\cos(x/2) \prod_{\nu=2}^{2n} \sin((x - x_\nu)/2)\Big) w(x)$ on $[-\pi, \pi)$. According to Theorem 2.4, such trigonometric polynomial has $2n$ distinct simple zeros in $[-\pi, \pi)$. Therefore, $\mathbf{y}^* \in S_{2n-1}$ and $\mathbf{y}^* = F_n(\mathbf{x})$. Since $\mathbf{y} = F_n(\mathbf{x})$, because of uniqueness we have $\mathbf{y}^* = \mathbf{y}$, i.e., the mapping F_n is continuous on $\overline{S}_{2n-1}$.

Now, we prove that the mapping F_n has a fixed point. The mapping F_n is continuous on the bounded, convex and closed set $\overline{S}_{2n-1} \subset \mathbb{R}^{2n-1}$. Applying the Brouwer fixed point theorem (see [14]) we conclude that there exists a fixed point of F_n. Since $F_n(\mathbf{x}) \in S_{2n-1}$ for all $\mathbf{x} \in \overline{S}_{2n-1} \backslash S_{2n-1}$, the fixed point of F_n belongs to S_{2n-1}.

If we denote the fixed point of F_n by $\mathbf{x} = (x_2, x_3, \ldots, x_{2n})$, then

$$\int_{-\pi}^{\pi} T_n(x) \cos kx\, p(T_n(x)) w(x)\,\mathrm{d}x = 0, \quad k = 0, 1, \ldots, n-1,$$
$$\int_{-\pi}^{\pi} T_n(x) \sin jx\, p(T_n(x)) w(x)\,\mathrm{d}x = 0, \quad j = 1, 2, \ldots, n-1,$$

where $T_n(x) = \cos(x/2) \prod_{\nu=2}^{2n} \sin((x - x_\nu)/2)$, and we get what is stated. □

Now, for $a \in [0, 1]$, $n \in \mathbb{N}$, and

$$F(\mathbf{x}, a) = \int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2as_1+1} \prod_{\nu=2}^{2n} \left|\sin\frac{x - x_\nu}{2}\right|^{2as_\nu+1} \operatorname{sgn}\left(\prod_{\nu=2}^{2n} \sin\frac{x - x_\nu}{2}\right) t_{n-1}(x)\, w(x)\,\mathrm{d}x, \quad t_{n-1} \in \mathcal{T}_{n-1}, \tag{16}$$

we consider the following problem

$$F(\mathbf{x}, a) = 0 \quad \text{for all } t_{n-1} \in \mathcal{T}_{n-1}, \tag{17}$$

with unknowns $x_2, x_3, \ldots, x_{2n}$.

The σ–orthogonality conditions (14) (with $x_1 = -\pi$) are equivalent to the problem (17) with $a = 1$. Therefore, the nodes of the quadrature rule (1) can be obtained as a solution of the problem (17) for $a = 1$.

From Theorem 4.2 we conclude that the problem (17) has solutions in the simplex S_{2n-1} for every $a \in [0, 1]$.

In Section 1 we proved that for $a = 0$ the problem (17) has the unique solution in the simplex S_{2n-1}, and, as we have already seen, the solution is also unique in the simplex S_{2n-1} for $a = 1$. Our aim is to prove the uniqueness of the solution $\mathbf{x} \in S_{2n-1}$ of the problem (17) for all $a \in (0, 1)$. For this purpose we use the mathematical induction on n.

Let us introduce the following notations:

$$W(\mathbf{x}, a, x) = \prod_{\nu=2}^{2n} \left|\sin \frac{x - x_\nu}{2}\right|^{2as_\nu+2}$$

and

$$\phi_k(\mathbf{x}, a) = \int_{-\pi}^{\pi} \left(\cos \frac{x}{2}\right)^{2as_1+1} \frac{W(\mathbf{x}, a, x)}{\sin \dfrac{x - x_k}{2}}\, w(x)\, \mathrm{d}x, \quad \text{for } k = 2, 3, \ldots, 2n. \tag{18}$$

Applying the same arguments as in [4, Lemma 3.2], one can prove the following auxiliary results.

Lemma 4.3. *There exists $\varepsilon > 0$ such that for every $a \in [0, 1]$ the solutions $\mathbf{x}$ of the problem (17) belong to the simplex $\overline{S}_\varepsilon = \{\mathbf{y} : \varepsilon \le y_2 + \pi, \varepsilon \le y_3 - y_2, \ldots, \varepsilon \le y_{2n} - y_{2n-1}, \varepsilon \le \pi - y_{2n}\}$.*

Lemma 4.4. *The problem (17) and the following problem*

$$\phi_k(\mathbf{x}, a) = 0, \quad k = 2, 3, \ldots, 2n, \tag{19}$$

where $\phi_k(\mathbf{x}, a)$ are given by (18), are equivalent in the simplex S_{2n-1}.

Proof. First we assume that $\mathbf{x} \in S_{2n-1}$ is a solution of the problem (17). Obviously,

$$\prod_{\substack{\nu=2\\ \nu\ne k}}^{2n} \sin \frac{x - x_\nu}{2} \in \mathcal{T}_{n-1}, \quad k = 2, 3, \ldots, 2n,$$

and for all $k = 2, 3, \ldots, 2n$ we have

$$\int_{-\pi}^{\pi} \left(\cos \frac{x}{2}\right)^{2as_1+1} \prod_{\nu=2}^{2n} \left|\sin \frac{x - x_\nu}{2}\right|^{2as_\nu+1} \operatorname{sgn}\left(\prod_{\nu=2}^{2n} \sin \frac{x - x_\nu}{2}\right) \prod_{\substack{\nu=2\\ \nu\ne k}}^{2n} \sin \frac{x - x_\nu}{2}\, w(x)\, \mathrm{d}x = 0,$$

i.e., $\mathbf{x}$ is a solution of (19).

Conversely, let $\mathbf{x}$ be a solution of (19). By using the fact that every trigonometric polynomial $t \in \mathcal{T}_{n-1}$ can be represented in the following way (see [4])

$$t(x) = \sum_{k=2}^{2n} t(x_k) \frac{\prod\limits_{\nu=2}^{2n} \sin \dfrac{x - x_\nu}{2}}{\sin \dfrac{x - x_k}{2} \prod\limits_{\substack{\nu=2\\ \nu\ne k}}^{2n} \sin \dfrac{x_k - x_\nu}{2}},$$

we have

$$\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\operatorname{sgn}\left(\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}\right)t(x)\,w(x)\,\mathrm{d}x$$

$$=\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\operatorname{sgn}\left(\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}\right)\sum_{k=2}^{2n}t(x_k)\frac{\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}}{\sin\frac{x-x_k}{2}\prod_{\substack{\nu=2\\ \nu\neq k}}^{2n}\sin\frac{x_k-x_\nu}{2}}\,w(x)\,\mathrm{d}x$$

$$=\sum_{k=2}^{2n}\frac{t(x_k)}{\prod_{\substack{\nu=2\\ \nu\neq k}}^{2n}\sin\frac{x_k-x_\nu}{2}}\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\operatorname{sgn}\left(\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}\right)\frac{\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}}{\sin\frac{x-x_k}{2}}\,w(x)\,\mathrm{d}x$$

$$=\sum_{k=2}^{2n}\frac{t(x_k)}{\prod_{\substack{\nu=2\\ \nu\neq k}}^{2n}\sin\frac{x_k-x_\nu}{2}}\phi_k(\mathbf{x},a)=0,$$

for all $t\in\mathcal{T}_{n-1}$. Therefore, $\mathbf{x}$ is a solution of the problem (17). □

According to Lemmas 4.3 and 4.4 we conclude that the problems (17) and (19) are equivalent in the simplex $\overline{S}_\varepsilon$, for some $\varepsilon>0$. Also, these problems are equivalent in the simplex $\overline{S}_{\varepsilon_1}$, for all $0<\varepsilon_1<\varepsilon$, but they are not equivalent in $\overline{S}_{2n-1}$.

By using the same arguments as in [9, Lemma 2.4], one can prove the following auxiliary results.

Lemma 4.5. *Let $p_{\xi,\eta}(x)$ be a continuous function on $[-\pi,\pi]$, which depends continuously on parameters $\xi,\eta\in[c,d]$. i.e., if (ξ_m,η_m) approaches (ξ_0,η_0) then the sequences $p_{\xi_m,\eta_m}(x)$ tends to $p_{\xi_0,\eta_0}(x)$ for every fixed x. If the solution $\mathbf{x}(\xi,\eta)$ of the problem* (19) *with the weight function $p_{\xi,\eta}(x)w(x)$ is always unique for every $(\xi,\eta)\in[c,d]^2$, then the solution $\mathbf{x}(\xi,\eta)$ depends continuously on $(\xi,\eta)\in[c,d]^2$.*

Now, we are going to prove the main theorem.

Theorem 4.6. *The problem* (19) *has a unique solution in the simplex S_{2n-1} for all $a\in[0,1]$.*

Proof. Let us call the problem (19) as $(a;s_2,s_3,\ldots,s_{2n};w)$ problem. We prove this assertion by mathematical induction on n.

The uniqueness for $n=0$ is trivial.

As an induction hypothesis, we suppose that the $(a;s_2,s_3,\ldots,s_{2n-2};w)$ problem has a unique solution for every $a\in[0,1]$ and for every weight function w, which are integrable and nonnegative on the interval $[-\pi,\pi)$, vanishing there only on a set of a measure zero.

For $(\xi,\eta)\in[-\pi,\pi]^2$ we define the weight functions

$$p_{\xi,\eta}(x)=\left|\sin\frac{x-\xi}{2}\right|^{2as_{2n-1}+2}\left|\sin\frac{x-\eta}{2}\right|^{2as_{2n}+2}w(x).$$

According to the induction hypothesis, for every $a\in[0,1]$ the $(a;s_2,s_3,\ldots,s_{2n-2};p_{\xi,\eta})$ problem has a unique solution $(x_2(\xi,\eta),x_3(\xi,\eta),\ldots,x_{2n-2}(\xi,\eta))$, such that $-\pi<x_2(\xi,\eta)<\cdots<x_{2n-2}(\xi,\eta)<\pi$, and, according to Lemma 4.5, $x_\nu(\xi,\eta)$ depends continuously on $(\xi,\eta)\in[-\pi,\pi]^2$, for all $\nu=2,3,\ldots,2n-2$.

We are going to prove that the solution of the $(a;s_2,\ldots,s_{2n};w)$ problem is unique for every $a\in[0,1]$.

Let us denote $x_{2n-1}(\xi,\eta)=\xi$, $x_{2n}(\xi,\eta)=\eta$,

$$W(\mathbf{x}(\xi,\eta),a,x)=\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu(\xi,\eta)}{2}\right|^{2as_\nu+2},$$

and

$$\phi_k(\mathbf{x}(\xi,\eta),a)=\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-x_k(\xi,\eta)}{2}}\,w(x)\,\mathrm{d}x,\quad k=2,3,\dots,2n.$$

Applying the induction hypothesis we obtain that

$$\phi_k(\mathbf{x}(\xi,\eta),a)=0,\quad k=2,3,\dots,2n-2, \tag{20}$$

for $(\xi,\eta)\in D$, where $D=\{(\xi,\eta): x_{2n-2}(\xi,\eta)<\xi<\eta<\pi\}$.

Let us now consider the following problem in D with unknown $\mathbf{t}=(\xi,\eta)$:

$$\begin{aligned}\varphi_1(\mathbf{t},a)&=\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-\xi}{2}}\,w(x)\,\mathrm{d}x=0,\\ \varphi_2(\mathbf{t},a)&=\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\frac{W(\mathbf{x}(\xi,\eta),a,x)}{\sin\dfrac{x-\eta}{2}}\,w(x)\,\mathrm{d}x=0.\end{aligned} \tag{21}$$

If $(\xi,\eta)\in D$ is a solution of the problem (21), then, according to (20), $\mathbf{x}(\xi,\eta)$ is a solution of the $(a;s_2,\dots,s_{2n};w)$ problem in the simplex S_{2n-1}. To the contrary, if $\mathbf{x}=(x_2,x_3,\dots,x_{2n})$ is a solution of the $(a;s_2,\dots,s_{2n};w)$ problem in the simplex S_{2n-1}, then (x_{2n-1},x_{2n}) is a solution of the problem (21). Applying Lemma 4.3 we conclude that every solution of the problem (21) belongs to $\overline{D}_\varepsilon=\{(\xi,\eta):\varepsilon\le\xi-x_{2n-2}(\xi,\eta),\varepsilon\le\eta-\xi,\varepsilon\le\pi-\eta\}$, for some $\varepsilon>0$.

Let $\mathbf{x}$ be a solution of the problem $(a;s_2,s_3,\dots,s_{2n};w)$ in $\overline{S}_\varepsilon$. Then, differentiating φ_1 with respect to the x_k, for all $k=2,3,\dots,2n-2,2n$, we have

$$\begin{aligned}\frac{\partial\varphi_1}{\partial x_k}=&-(as_k+1)\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\operatorname{sgn}\left(\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}\right)\\&\times\prod_{\substack{\nu=2\\ \nu\neq k,\,\nu\neq 2n-1}}^{2n}\sin\frac{x-x_\nu}{2}\cos\frac{x-x_k}{2}\,w(x)\,\mathrm{d}x.\end{aligned}$$

Since

$$\prod_{\substack{\nu=2\\ \nu\neq k,\,\nu\neq 2n-1}}^{2n}\sin\frac{x-x_\nu}{2}\cos\frac{x-x_k}{2}\in\mathcal{T}_{n-1},$$

applying Lemma 4.4, we conclude that $\dfrac{\partial\varphi_1}{\partial x_k}=0$, for all $k=2,3,\dots,2n-2,2n$. Further, by using the following simple equality

$$\cos\frac{x-y}{2}=\cos\frac{x}{2}\cos\frac{y}{2}+\sin\frac{x-y}{2}\cos\frac{y}{2}\sin\frac{y}{2}+\cos\frac{x-y}{2}\sin^2\frac{y}{2},$$

for

$$I=\frac{\partial\varphi_1}{\partial x_{2n-1}}=-\frac{2as_{2n-1}+1}{2}\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\substack{\nu=2\\ \nu\neq 2n-1}}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2}\left|\sin\frac{x-x_{2n-1}}{2}\right|^{2as_{2n-1}}\cos\frac{x-x_{2n-1}}{2}\,w(x)\,\mathrm{d}x,$$

we obtain

$$\cos\frac{x_{2n-1}}{2}I=-\frac{2as_{2n-1}+1}{2}\left(I_1+\sin\frac{x_{2n-1}}{2}I_2\right), \tag{22}$$

where

$$I_1 = \int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2as_1+1} \prod_{\substack{\nu=2\\ \nu\neq 2n-1}}^{2n} \left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2} \left|\sin\frac{x-x_{2n-1}}{2}\right|^{2as_{2n-1}} \cos\frac{x}{2}\, w(x)\,\mathrm{d}x$$

and

$$I_2 = \int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2as_1+1} \prod_{\substack{\nu=2\\ \nu\neq 2n-1}}^{2n} \left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2} \left|\sin\frac{x-x_{2n-1}}{2}\right|^{2as_{2n-1}} \sin\frac{x-x_{2n-1}}{2}\, w(x)\,\mathrm{d}x.$$

According to (21) we have that $I_2 = 0$, while $I_1 > 0$ (the integrand does not change its sign on $[-\pi,\pi)$). Since $-\pi < x_{2n-1} < \pi$, it follows from (22) that

$$\operatorname{sgn}\left(\frac{\partial\varphi_1}{\partial x_{2n-1}}\right) = -1.$$

Analogously,

$$\frac{\partial\varphi_2}{\partial x_k} = 0, \quad k = 2,3,\ldots,2n-1, \quad \operatorname{sgn}\left(\frac{\partial\varphi_2}{\partial x_{2n}}\right) = -1.$$

Thus, at any solution $\mathbf{t} = (\xi,\eta)$ of the problem (21) from $\overline{D}_\varepsilon$ we have

$$\frac{\partial\varphi_1}{\partial\xi} = \sum_{\substack{k=2\\ k\neq 2n-1}}^{2n} \frac{\partial\varphi_1}{\partial x_k}\cdot\frac{\partial x_k}{\partial\xi} + \frac{\partial\varphi_1}{\partial x_{2n-1}} = \frac{\partial\varphi_1}{\partial x_{2n-1}}, \quad \frac{\partial\varphi_1}{\partial\eta} = 0,$$

$$\frac{\partial\varphi_2}{\partial\eta} = \sum_{k=2}^{2n-1} \frac{\partial\varphi_2}{\partial x_k}\cdot\frac{\partial x_k}{\partial\eta} + \frac{\partial\varphi_2}{\partial x_{2n}} = \frac{\partial\varphi_2}{\partial x_{2n}}, \quad \frac{\partial\varphi_2}{\partial\xi} = 0,$$

so, the determinant of the corresponding Jacobian $J(\mathbf{t})$ is positive.

In order to finish the proof, we have to prove that the problem (21) has a unique solution in $\overline{D}_{\varepsilon/2}$. We do it by using the topological degree of a mapping. Let us define the mapping $\boldsymbol{\varphi}(\mathbf{t},a) : \overline{D}_{\varepsilon/2}\times[0,1] \to \mathbb{R}^2$, by $\boldsymbol{\varphi}(\mathbf{t},a) = (\varphi_1(\mathbf{t},a),\varphi_2(\mathbf{t},a))$, $\mathbf{t}\in\mathbb{R}^2$, $a\in[0,1]$. If $\mathbf{x}(\xi,\eta)$ is a solution of the $(a;s_2,\ldots,s_{2n};w)$ problem in $\overline{S}_\varepsilon$, then the solutions of the problem $\boldsymbol{\varphi}(\mathbf{t},a) = (0,0)$ in $\overline{D}_{\varepsilon/2}$ belong to $\overline{D}_\varepsilon$. The problem $\boldsymbol{\varphi}(\mathbf{t},a) = (0,0)$ has the unique solution in $\overline{D}_{\varepsilon/2}$ for $a = 0$. It is obvious that $\boldsymbol{\varphi}(\,\cdot\,,0)$ is differentiable on $D_{\varepsilon/2}$, the mapping $\boldsymbol{\varphi}(\mathbf{t},a)$ is continuous in $\overline{D}_{\varepsilon/2}\times[0,1]$, and $\boldsymbol{\varphi}(\mathbf{t},a)\neq(0,0)$ for all $\mathbf{t}\in\partial D_{\varepsilon/2}$ and $a\in[0,1]$. Then $\deg(\boldsymbol{\varphi}(\,\cdot\,,a), D_{\varepsilon/2},(0,0)) = \operatorname{sgn}(\det(J(\mathbf{t}))) = 1$, for all $a\in[0,1]$, i.e., $\deg(\boldsymbol{\varphi}(\,\cdot\,,a),D_{\varepsilon/2},(0,0))$ is a constant independent of a. Thus, the problem $\boldsymbol{\varphi}(\mathbf{t},a) = (0,0)$ has the unique solution in $D_{\varepsilon/2}$ for all $a\in[0,1]$, which means that the $(a;s_2,\ldots,s_{2n};w)$ problem has a unique solution in S_{2n-1} which belongs to $\overline{S}_\varepsilon$. □

Theorem 4.7. *The solution* $\mathbf{x} = \mathbf{x}(a)$ *of the problem* (19) *depends continuously on* $a\in[0,1]$.

Proof. Let us suppose that $\{a_m\}$, $a_m\in[0,1]$, $m\in\mathbb{N}$, is a convergent sequence, which converges to $a^*\in[0,1]$. Then for every a_m, $m\in\mathbb{N}$, there exists the unique solution $\mathbf{x}(a_m) = (x_2(a_m),\ldots,x_{2n}(a_m))$ of the system (19) with $a = a_m$. Let $\mathbf{x}(a^*) = (x_2(a^*),\ldots,x_{2n}(a^*))$ be the unique solution of system (19) for $a = a^*$. Let $\mathbf{x}^* = (x_2^*,\ldots,x_{2n}^*)$ be an arbitrary limit point of the sequence $\mathbf{x}(a_m)$ when $a_m\to a^*$. Then, according to Theorem 4.6, for each $m\in\mathbb{N}$ we obtain

$$\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2a_ms_1+1} \frac{W(\mathbf{x}(a_m),a_m,x)}{\sin\dfrac{x-x_k(a_m)}{2}}\, w(x)\,\mathrm{d}x = 0, \quad k = 2,3,\ldots,2n.$$

When $a_m \to a^*$ the above equations lead to

$$\int_{-\pi}^{\pi} \left(\cos\frac{x}{2}\right)^{2a^* s_1+1} \frac{W(\mathbf{x}^*, a^*, x)}{\sin\dfrac{x-x_k^*}{2}} w(x)\,\mathrm{d}x = 0, \quad k = 2,3,\ldots,2n.$$

Hence, according to Theorem 4.6, we have that $\mathbf{x}^* = \mathbf{x}(a^*)$, i.e., $\lim\limits_{a_m\to a^*} \mathbf{x}(a_m) = \mathbf{x}(a^*)$, which gives what is claimed. □

5. Numerical construction of quadrature rules

In this section we present one method for construction of quadrature rules of the form (1), based on the theoretical results given in the previous sections. So, we chose $x_1 = -\pi$ and obtain the nodes $x_2, x_3, \cdots, x_{2n}$ by solving problem (19) for $a = 1$. Since that problem is a system of nonlinear equations, Newton–Kantorovič method can be applied. The theoretical results obtained in Section 4 suggest that for fixed n the system (19) can be solved progressively, for an increasing sequence of values for a, up to the value $a = 1$. We use the solution for some $a^{(i)}$ as the initial iteration in Newton–Kantorovič method for calculating the solution for $a^{(i+1)}$, $a^{(i)} < a^{(i+1)} \leq 1$. If for some chosen $a^{(i+1)}$ Newton–Kantorovič method does not converge, we decrease $a^{(i+1)}$ such that it becomes convergent, which is always possible according to Theorem 4.7. Practically, we can in each step set $a^{(i+1)} = 1$, and, if that iterative process is not convergent, we set $a^{(i+1)} := (a^{(i+1)} + a^{(i)})/2$ until it becomes convergent. As the initial iteration for the first iterative process, for some $a^{(1)} > 0$, we choose the zeros of the corresponding orthogonal trigonometric polynomial of degree n, i.e., the solution of system (19) for $a = 0$.

Let us introduce the following matrix notation

$$\mathbf{x} = \begin{bmatrix} x_2 & x_3 & \cdots & x_{2n} \end{bmatrix}^T, \quad \mathbf{x}^{(m)} = \begin{bmatrix} x_2^{(m)} & x_3^{(m)} & \cdots & x_{2n}^{(m)} \end{bmatrix}^T, \quad m = 0,1,\ldots,$$

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} \phi_2(\mathbf{x}) & \phi_3(\mathbf{x}) & \cdots & \phi_{2n}(\mathbf{x}) \end{bmatrix}^T.$$

By

$$\mathbf{W} = \mathbf{W}(\mathbf{x}) = [w_{i,j}]_{(2n-1)\times(2n-1)} = \left[\frac{\partial \phi_{i+1}}{\partial x_{j+1}}\right]_{(2n-1)\times(2n-1)}$$

we denote Jacobian of $\boldsymbol{\phi}(\mathbf{x})$, with entries given by

$$\frac{\partial \phi_i}{\partial x_j} = -(1+as_j)\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\nu=2}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+1}\operatorname{sgn}\left(\prod_{\nu=2}^{2n}\sin\frac{x-x_\nu}{2}\right)\prod_{\substack{\nu=2\\ \nu\neq i,\nu\neq j}}^{2n}\sin\frac{x-x_\nu}{2}\cos\frac{x-x_j}{2}\,w(x)\,\mathrm{d}x,$$

for $i \neq j$, $i,j = 2,3,\ldots,2n$, and

$$\frac{\partial \phi_i}{\partial x_i} = -\frac{1+2as_i}{2}\int_{-\pi}^{\pi}\left(\cos\frac{x}{2}\right)^{2as_1+1}\prod_{\substack{\nu=2\\ \nu\neq i}}^{2n}\left|\sin\frac{x-x_\nu}{2}\right|^{2as_\nu+2}\left|\sin\frac{x-x_i}{2}\right|^{2as_i}\cos\frac{x-x_i}{2}\,w(x)\,\mathrm{d}x, \quad i = 2,3,\ldots,2n.$$

All of the above integrals can be computed by using a Gaussian type quadrature rule for trigonometric polynomials.

The Newton–Kantorovič method for calculating the zeros of the σ–orthogonal trigonometric polynomial $T_{\sigma,n}$ is given as follows

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} - \mathbf{W}^{-1}(\mathbf{x}^{(m)})\boldsymbol{\phi}(\mathbf{x}^{(m)}), \quad m = 0,1,\ldots,$$

and for sufficiently good chosen initial approximation $\mathbf{x}^{(0)}$, it has the quadratic convergence.

When the nodes of quadrature rule (1) are known, it is possible to calculate the corresponding weights. They can be calculating by using the Hermite trigonometric interpolation polynomial (see [4, 5]), but it is very difficult. Our method is based on the method given in [7] for construction of Gauss–Turán quadrature rules (for algebraic polynomials) and generalized for Chakalov–Popoviciu's type quadrature rules in [11, 12] (for algebraic polynomials, too). An adaptation of that method to the construction of quadrature rules with the maximal trigonometric degree of exactness with an odd number of multiple nodes was proposed in [9]. Here we use the similar adaptation for the construction of quadrature rule (1) with an even number of multiple nodes.

We use the facts that quadrature rule (1) is of interpolatory type and that it has the maximal trigonometric degree of exactness, i.e., that it is exact for all trigonometric polynomials of degree less than or equal to $\sum_{\nu=1}^{2n}(s_\nu+1)-1=\sum_{\nu=1}^{2n}s_\nu+2n-1$. So, the weights can be calculated requiring that quadrature rule (1) integrates exactly all trigonometric polynomials of degree less than or equal to $\sum_{\nu=1}^{2n}s_\nu+n-1$, and in addition $\cos\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$, when $\sum_{\nu=1}^{2n}(2s_\nu+1)x_\nu=\ell\pi$ for an odd $\ell\in\mathbb{Z}$, or $\sin\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$ when $\sum_{\nu=1}^{2n}(2s_\nu+1)x_\nu=\ell\pi$ for an even $\ell\in\mathbb{Z}$. If $\sum_{\nu=1}^{2n}(2s_\nu+1)x_\nu\neq\ell\pi$, $\ell\in\mathbb{Z}$, one can choose to require exactness for $\cos\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$ or $\sin\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$ arbitrary. In such a way we obtain a system of linear equations for the unknown weights. That system can be solved by decomposing into a set of $2n$ upper triangular systems. In what follows we present that method of decomposing in details in the case when $\sum_{\nu=1}^{2n}(2s_\nu+1)x_\nu=\ell\pi$, $\ell\in\mathbb{Z}$.

Let us denote

$$\Omega_\nu(x)=\prod_{\substack{i=1\\ i\neq\nu}}^{2n}\left(\sin\frac{x-x_i}{2}\right)^{2s_i+1},\ \nu=1,2,\ldots,2n,\quad u_{k,\nu}(x)=\left(\sin\frac{x-x_\nu}{2}\right)^{k}\Omega_\nu(x),\ \nu=1,2,\ldots,2n,\ k=0,1,\ldots,2s_\nu,$$

and

$$t_{k,\nu}(x)=\begin{cases} u_{k,\nu}(x)\cos\dfrac{x-x_\nu}{2}, & k\text{–even},\\ u_{k,\nu}(x), & k\text{–odd},\end{cases}\qquad \nu=1,2,\ldots,2n,\ k=0,1,\ldots,2s_\nu. \tag{23}$$

Since $t_{k,\nu}$ is a trigonometric polynomial of degree

$$\frac{1}{2}\sum_{\substack{i=1\\ i\neq\nu}}^{2n}(2s_i+1)+\left[\frac{k}{2}\right]+\frac{1}{2}\le\frac{1}{2}\sum_{\substack{i=1\\ i\neq\nu}}^{2n}(2s_i+1)+s_\nu+\frac{1}{2}=\sum_{\nu=1}^{2n}s_\nu+n,$$

which has the leading term $\cos\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$ or $\sin\Big(\sum_{\nu=1}^{2n}s_\nu+n\Big)x$ (but not the both of them), the quadrature rule is exact for $t_{k,\nu}$, i.e., $R_n(t_{k,\nu})=0$ for all $\nu=1,2,\ldots,2n$, $k=0,1,\ldots,2s_\nu$. Since for all $i\neq\nu$ we have that $t_{k,\nu}^{(j)}(x_i)=0$, $j=0,1,\ldots,2s_j$, then

$$\mu_{k,\nu}=\int_{-\pi}^{\pi}t_{k,\nu}(x)\,w(x)\,\mathrm{d}x=\sum_{j=0}^{2s_\nu}A_{j,\nu}t_{k,\nu}^{(j)}(x_\nu),\quad \nu=1,2,\ldots,2n. \tag{24}$$

In such a way, we obtain $2n$ independent systems for calculating the weights $A_{j,\nu}$, $j=0,1,\ldots,s_{2\nu}$, $\nu=1,2,\ldots,2n$. Here, we need to calculate the derivatives $t_{k,\nu}^{(j)}(x)$, $k=0,1,\ldots,2s_\nu$, $j=0,1,\ldots,2s_\nu$, and for that we use the following result (the proof can be obtained by the similar arguments as in [9]).

Lemma 5.1. *For the trigonometric polynomials $t_{k,\nu}$, given by* (23) *we have*

$$t_{k,\nu}^{(i)}(x_\nu)=0,\quad i<k;\quad t_{k,\nu}^{(k)}(x_\nu)=\frac{k!}{2^k}\Omega_\nu(x_\nu),$$

and for $i > k$

$$t_{k,\nu}^{(i)}(x_\nu) = \begin{cases} v_{k,\nu}^{(i)}(x_\nu), & k\text{-even}, \\ u_{k,\nu}^{(i)}(x_\nu), & k\text{-odd}, \end{cases} \qquad \text{where} \quad v_{k,\nu}^{(i)}(x_\nu) = \sum_{m=0}^{[i/2]} \binom{i}{2m} \frac{(-1)^m}{2^{2m}} u_{k,\nu}^{(i-2m)}(x_\nu),$$

and the sequence $u_{k,\nu}^{(i)}(x_\nu)$, $k \in \mathbb{N}_0$, $i \in \mathbb{N}_0$, is the solution of the difference equation

$$f_{k,\nu}^{(i)} = \sum_{m=[(i-k)/2]}^{[(i-1)/2]} \binom{i}{2m+1} \frac{(-1)^m}{2^{2m+1}} f_{k-1,\nu}^{(i-2m-1)}(x_\nu), \quad \nu = 1,2,\ldots,2n, \quad k \in \mathbb{N}, \quad i \in \mathbb{N}_0,$$

with the initial conditions $f_{0,\nu}^{(i)} = \Omega_\nu^{(i)}(x_\nu)$.

The calculating of $\Omega_\nu^{(i)}(x_\nu)$, $i \in \mathbb{N}_0$, $\nu = 1,2,\ldots,2n$, can be done as in [9, Lemma 3.3].

Finally, Lemma 5.1 implies that systems (24) are the following upper triangular systems

$$\begin{bmatrix} t_{0,\nu}(x_\nu) & t'_{0,\nu}(x_\nu) & \cdots & t_{0,\nu}^{(2s_\nu)}(x_\nu) \\ & t'_{1,\nu}(x_\nu) & \cdots & t_{1,\nu}^{(2s_\nu)}(x_\nu) \\ & & \ddots & \vdots \\ & & & t_{2s_\nu,\nu}^{(2s_\nu)}(x_\nu) \end{bmatrix} \cdot \begin{bmatrix} A_{0,\nu} \\ A_{1,\nu} \\ \vdots \\ A_{2s_\nu,\nu} \end{bmatrix} = \begin{bmatrix} \mu_{0,\nu} \\ \mu_{1,\nu} \\ \vdots \\ \mu_{2s_\nu,\nu} \end{bmatrix}, \quad \nu = 1,2\ldots,2n.$$

Remark 5.2. *If $\sum_{\nu=1}^{2n}(2s_\nu + 1)x_\nu \neq \ell\pi$, $\ell \in \mathbb{Z}$, then in the case when k is even we choose $t_{k,\nu}$, $k = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,2n$, in one of the following ways*

$$t_{k,\nu}(x) = u_{k,\nu}(x) \cos \frac{x + \sum\limits_{\substack{i=1 \\ i\neq\nu}}^{2n} (2s_i + 1)x_i + 2s_\nu x_\nu}{2} \quad \text{or} \quad t_{k,\nu}(x) = u_{k,\nu}(x) \sin \frac{x + \sum\limits_{\substack{i=1 \\ i\neq\nu}}^{2n} (2s_i + 1)x_i + 2s_\nu x_\nu}{2},$$

which provides that

$$t_{k,\nu} \in \mathcal{T}_{\sum_{\nu=1}^{2n} s_\nu + n} \ominus \operatorname{span}\left\{\cos\left(\sum_{\nu=1}^{2n} s_\nu + n\right)x\right\} \quad \text{or} \quad t_{k,\nu} \in \mathcal{T}_{\sum_{\nu=1}^{2n} s_\nu + n} \ominus \operatorname{span}\left\{\sin\left(\sum_{\nu=1}^{2n} s_\nu + n\right)x\right\},$$

$k = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,2n$, respectively.

Now we present one example as an illustration of the obtained theoretical results.

Example 5.3. *We choose weight function $w(x) = 1 + \cos 2x$, $x \in [-\pi,\pi)$, $n = 3$ and $\sigma = (3,3,3,4,4,4)$. For calculation of nodes we use described procedure, and we have two iterative processes: for $a = 1/2$ (with 9 iterations) and for $a = 1$ (with 8 iterations). Obtained numerical values for the nodes x_ν, $\nu = 1,2,\ldots,2n$, are the following:* −3.141592653589793, −2.264556388673865, −1.179320242581565, −0.1955027724705077, 0.8612188670819011, 2.178685249095223.

The corresponding weight coefficients $A_{j,\nu}$, $j = 0,1,\ldots,2s_\nu$, $\nu = 1,2,\ldots,6$, are given in Table 1 (numbers in parentheses denote decimal exponents).

j	$A_{j,1}$	$A_{j,2}$	$A_{j,3}$
0	1.676594372192496	7.305332409592605(−1)	3.487838013502532(−1)
1	−3.606295932640399(−3)	−8.705060748567067(−2)	6.802235520408829(−2)
2	4.450266475268382(−2)	1.929684165460602(−2)	1.120920249826093(−2)
3	−6.813768854008439(−2)	−9.825070962211830(−4)	8.081866652580276(−4)
4	2.575834718221877(−4)	1.050037121874394(−4)	6.594073161399015(−5)
5	−2.025745530496365(−7)	−2.102329131299247(−6)	1.808851807084914(−6)
6	3.780621226622374(−7)	1.413143818423462(−7)	9.154154335491590(−8)
j	$A_{j,4}$	$A_{j,5}$	$A_{j,6}$
0	1.880646586862865	9.170128528217915(−1)	7.296144529929201(−1)
1	6.328640684710125(−2)	−1.557489854117211(−1)	1.399304206896031(−1)
2	7.316493158593520(−2)	3.683058320865013(−2)	2.916174720939193(−2)
3	1.252054136886738(−3)	−2.974661584717919(−3)	2.588432069654582(−3)
4	7.029326017474412(−4)	3.381006758789474(−4)	2.606855435340879(−4)
5	6.219410600844099(−6)	−1.428080829445919(−5)	1.209621855269665(−5)
6	2.280678345497009(−6)	1.008832884300956(−6)	7.540115736887533(−7)
7	8.397304312672816(−9)	−1.865855244589502(−8)	1.546523556082379(−8)
8	2.276058254314503(−9)	8.981189180630658(−10)	6.501031023479647(−10)

Table 1: Weight coefficients $A_{j,\nu}$, $j = 0, 1, \ldots, 2s_\nu$, $\nu = 1, 2, \ldots, 6$, for $w(x) = 1 + \cos 2x$, $x \in [-\pi, \pi)$, $n = 3$ and $\sigma = (3, 3, 3, 4, 4, 4)$.

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