

LAPLACIAN ENERGY OF UNION AND CARTESIAN PRODUCT AND LAPLACIAN EQUIENERGETIC GRAPHS

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ABSTRACT. The Laplacian energy of a graph G with n vertices and m edges is defined as $LE(G) = \sum_{i=1}^n |\mu_i - 2m/n|$, where $\mu_1, \mu_2, \dots, \mu_n$ are the Laplacian eigenvalues of G . If two graphs G_1 and G_2 have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. Otherwise, this identity is violated. We determine a term Ξ , such that $LE(G_1) + LE(G_2) - \Xi \leq LE(G_1 \cup G_2) \leq LE(G_1) + LE(G_2) + \Xi$ holds for all graphs. Further, by calculating LE of the Cartesian product of some graphs, we construct new classes of Laplacian non-cospectral, Laplacian equienergetic graphs.

1. INTRODUCTION

Let G be a finite, simple, undirected graph with n vertices v_1, v_2, \dots, v_n and m edges. In what follows, we say that G is an (n, m) -graph. Let $A(G)$ be the adjacency matrix of G and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues.

Let $D(G)$ be the diagonal matrix whose (i, i) -th entry is the degree of a vertex v_i . The matrix $C(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G . The *Laplacian polynomial* of G is defined as $\psi(G, \mu) = \det[\mu I - C(G)]$, where I is an identity matrix. The eigenvalues of $C(G)$, denoted by $\mu_i = \mu_i(G)$, $i = 1, 2, \dots, n$, are called the *Laplacian eigenvalues* of G [16]. Two graphs are said to be *Laplacian cospectral* if they have same Laplacian eigenvalues. The adjacency eigenvalues and Laplacian eigenvalues satisfies the following conditions:

$$\sum_{i=1}^n \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^n \mu_i = 2m.$$

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The *energy* of a graph G is defined as

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|.$$

It was introduced by one of the present authors in the 1970s, and since then has been much studied in both chemical and mathematical literature. For details see the book [15] and the references cited therein.

The *Laplacian energy* of a graph was introduced a few years ago [13] and is defined as

$$LE(G) = \sum_{i=1}^n \left| \mu_i(G) - \frac{2m}{n} \right|.$$

This definition is chosen so as to preserve the main features of the ordinary graph energy \mathcal{E} , see [18]. Basic properties and other results on Laplacian energy can be found in the survey [1], the recent papers [6–8, 11, 12, 17, 19], and the references cited therein.

2. LAPLACIAN ENERGY OF UNION OF GRAPHS

Let G_1 and G_2 be two graphs with disjoint vertex sets. Let for $i = 1, 2$, the vertex and edges sets of G_i be, respectively, V_i and E_i . The *union* of G_1 and G_2 is a graph $G_1 \cup G_2$ with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. If G_1 is an (n_1, m_1) -graph and G_2 is an (n_2, m_2) -graph then $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges. It is easy to see that the Laplacian spectrum of $G_1 \cup G_2$ is the union of the Laplacian spectra of G_1 and G_2 .

In [13] it was proven that if G_1 and G_2 have equal average vertex degrees, then $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$. If the average vertex degrees are not equal, that is $\frac{2m_1}{n_1} \neq \frac{2m_2}{n_2}$, then it may be either $LE(G_1 \cup G_2) > LE(G_1) + LE(G_2)$ or $LE(G_1 \cup G_2) < LE(G_1) + LE(G_2)$ or, exceptionally, $LE(G_1 \cup G_2) = LE(G_1) + LE(G_2)$ [13].

In this section we study the Laplacian establish some additional relations between $LE(G_1 \cup G_2)$ and $LE(G_1) + LE(G_2)$.

Theorem 2.1. *Let G_1 be an (n_1, m_1) -graph and G_2 be an (n_2, m_2) -graph, such that $\frac{2m_1}{n_1} > \frac{2m_2}{n_2}$. Then*

$$\begin{aligned} LE(G_1) + LE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} &\leq LE(G_1 \cup G_2) \\ (2.1) \qquad \qquad \qquad &\leq LE(G_1) + LE(G_2) + \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2}. \end{aligned}$$

Proof. For the sake of simplicity, denote $G_1 \cup G_2$ by G . Then G is an $(n_1 + n_2, m_1 + m_2)$ -graphs. By the definition of Laplacian energy,

$$\begin{aligned}
LE(G_1 \cup G_2) &= \sum_{i=1}^{n_1+n_2} \left| \mu_i(G) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| + \sum_{i=n_1+1}^{n_1+n_2} \left| \mu_i(G) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
&= \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} + \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
&\quad + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} + \frac{2m_2}{n_2} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
&\leq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| + n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right| \\
(2.2) \quad &\quad + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} \right| + n_2 \left| \frac{2m_2}{n_2} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|.
\end{aligned}$$

Since $n_2 m_1 > n_1 m_2$, Eq. (2.2) becomes

$$\begin{aligned}
LE(G_1 \cup G_2) &\leq LE(G_1) + n_1 \left(\frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \\
&\quad + LE(G_2) + n_2 \left(-\frac{2m_2}{n_2} + \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \\
&= LE(G_1) + LE(G_2) + \frac{4(n_2 m_1 - n_1 m_2)}{n_1 + n_2}
\end{aligned}$$

which is an upper bound.

To obtain the lower bound, we just have to note that in full analogy to the above arguments,

$$LE(G_1 \cup G_2) \geq \sum_{i=1}^{n_1} \left| \mu_i(G_1) - \frac{2m_1}{n_1} \right| - n_1 \left| \frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|$$

$$(2.3) \quad + \sum_{i=1}^{n_2} \left| \mu_i(G_2) - \frac{2m_2}{n_2} \right| - n_2 \left| \frac{2m_2}{n_2} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right|.$$

Since $n_2m_1 > n_1m_2$, the Eq. (2.3) becomes

$$\begin{aligned} LE(G_1 \cup G_2) &\geq LE(G_1) - n_1 \left(\frac{2m_1}{n_1} - \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \\ &\quad + LE(G_2) - n_2 \left(-\frac{2m_2}{n_2} + \frac{2(m_1 + m_2)}{n_1 + n_2} \right) \\ &= LE(G_1) + LE(G_2) - \frac{4(n_2m_1 - n_1m_2)}{n_1 + n_2} \end{aligned}$$

which is a lower bound. \square

Corollary 2.1. [13] *Let G_1 be an (n_1, m_1) -graph and G_2 be (n_2, m_2) -graph such that $\frac{2m_1}{n_1} = \frac{2m_2}{n_2}$. Then*

$$LE(G_1 \cup G_2) = LE(G_1) + LE(G_2).$$

Corollary 2.2. *Let G_1 be an r_1 -regular graph on n_1 vertices and G_2 be an r_2 -regular graph on n_2 vertices, such that $r_1 > r_2$. Then*

$$\begin{aligned} LE(G_1) + LE(G_2) - \frac{2n_1n_2(r_1 - r_2)}{n_1 + n_2} &\leq LE(G_1 \cup G_2) \\ &\leq LE(G_1) + LE(G_2) + \frac{2n_1n_2(r_1 - r_2)}{n_1 + n_2}. \end{aligned}$$

Proof. Result follows by setting $m_1 = n_1r_1/2$ and $m_2 = n_2r_2/2$ into Theorem 2.1. \square

Theorem 2.2. *Let G be an (n, m) -graph and \bar{G} be its complement, and let $m > n(n-1)/4$. Then*

$$LE(G) + LE(\bar{G}) - [4m - n(n-1)] \leq LE(G \cup \bar{G}) \leq LE(G) + LE(\bar{G}) + [4m - n(n-1)].$$

Proof. \bar{G} is a graph with n vertices and $n(n-1)/2 - m$ edges. Substituting this into Eq. (2.1), the result follows. \square

Theorem 2.3. *Let G be an (n, m) -graph and G' be the graph obtained from G by removing k edges, $0 \leq k \leq m$. Then*

$$LE(G) + LE(G') - 2k \leq LE(G \cup G') \leq LE(G) + LE(G') + 2k.$$

Proof. The number of vertices of G' is n and the number of edges is $m - k$. Substituting this in Eq. (2.1), the result follows. \square

3. LAPLACIAN ENERGY OF CARTESIAN PRODUCT

Let G be a graph with vertex set V_1 and H be a graph with vertex set V_2 . The Cartesian product of G and H , denoted by $G \times H$ is a graph with vertex set $V_1 \times V_2$, such that two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H or $v_1 = v_2$ and u_1 is adjacent to u_2 in G [14].

Lemma 3.1. [9] Let $A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Theorem 3.1. If $\mu_1, \mu_2, \dots, \mu_n$ are the Laplacian eigenvalues of a graph G , then the Laplacian eigenvalues of $G \times K_2$ are $\mu_1, \mu_2, \dots, \mu_n$ and $\mu_1 + 2, \mu_2 + 2, \dots, \mu_n + 2$.

Proof. The Laplacian matrix of $G \times K_2$ is

$$C(G \times K_2) = \begin{bmatrix} C(G) + I & -I \\ -I & C(G) + I \end{bmatrix} = \begin{bmatrix} C_0 & C_1 \\ C_1 & C_0 \end{bmatrix}$$

where $C(G)$ is the Laplacian matrix of G and I is an identity matrix of order n . By Lemma 3.1, the Laplacian spectrum of $G \times K_2$ is the union of the spectra of $C_0 + C_1$ and $C_0 - C_1$.

Here $C_0 + C_1 = C(G)$. Therefore, the eigenvalues of $C_0 + C_1$ are the Laplacian eigenvalues of G .

Because $C_0 - C_1 = C(G) + 2I$, the characteristic polynomial of $C_0 - C_1$ is

$$\begin{aligned} \psi(C_0 - C_1, \mu) &= \det [\mu I - (C_0 - C_1)] = \det [\mu I - (C(G) + 2I)] \\ &= \det [(\mu - 2)I - C(G)] = \psi(G, \mu - 2). \end{aligned}$$

Therefore the eigenvalues of $C_0 - C_1$ are $\mu_i + 2, i = 1, 2, \dots, n$. □

The Laplacian eigenvalues of the complete graph K_n are n ($n - 1$ times) and 0. The Laplacian eigenvalues of the complete bipartite regular graph $K_{k,k}$ are $2k, k$ ($2k - 2$ times) and 0. The Laplacian eigenvalues of the cocktail party graph $CP(k)$ (the regular graph on $n = 2k$ vertices and of degree $2k - 2$) are $2k$ ($k - 1$ times), $2k - 2$ (k times) and 0 [16]. Applying Theorem 3.1, we directly arrive at the following example.

Example 3.1.

$$LE(K_n \times K_2) = \begin{cases} 4n - 4, & \text{if } n > 2, \\ 2n, & \text{if } n \leq 2, \end{cases}$$

$$LE(K_{k,k} \times K_2) = \begin{cases} 8k - 4, & \text{if } k > 1, \\ 6k - 2, & \text{if } k = 1, \end{cases}$$

$$LE(CP(k) \times K_2) = 10k - 8, \quad \text{if } k \geq 2.$$

Theorem 3.2. Let G be an (n, m) -graph. Then

$$2[LE(G) - n] \leq LE(G \times K_2) \leq 2[LE(G) + n].$$

Proof. Let $\mu_1, \mu_2, \dots, \mu_n$ be the Laplacian eigenvalues of G . Then by Theorem 3.1, the Laplacian eigenvalues of $G \times K_2$ are $\mu_i, i = 1, 2, \dots, n$ and $\mu_i + 2, i = 1, 2, \dots, n$. The graph $G \times K_2$ has $2n$ vertices and $2m + n$ edges. Therefore,

$$\begin{aligned} LE(G \times K_2) &= \sum_{i=1}^n \left| \mu_i - \frac{2(2m+n)}{2n} \right| + \sum_{i=1}^n \left| \mu_i + 2 - \frac{2(2m+n)}{2n} \right| \\ (3.1) \qquad &= \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} - 1 \right| + \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} + 1 \right|. \end{aligned}$$

Equation (3.1) can be rewritten as

$$LE(G \times K_2) \leq \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right| + n + \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right| + n = 2LE(G) + 2n$$

which is an upper bound.

For lower bound, Eq. (3.1) can be rewritten as

$$LE(G \times K_2) \geq \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right| - n + \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right| - n = 2LE(G) - 2n.$$

□

Theorem 3.3. For a graph G with n vertices, $LE(G \times K_2) \geq 2n$.

Proof. From Eq. (3.1)

$$\begin{aligned} LE(G \times K_2) &\geq \left| \sum_{i=1}^n \left(\mu_i - \frac{2m}{n} - 1 \right) \right| + \left| \sum_{i=1}^n \left(\mu_i - \frac{2m}{n} + 1 \right) \right| \\ &= |2m - 2m - n| + |2m - 2m + n| = 2. \end{aligned}$$

□

4. LAPLACIAN EQUIENERGETIC GRAPHS

Two graphs G_1 and G_2 are said to be *equienergetic* if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$ [2]. For details see the book [15]. In analogy to this, two graphs G_1 and G_2 are said to be *Laplacian equienergetic* if $LE(G_1) = LE(G_2)$.

Obviously Laplacian cospectral graphs are Laplacian equienergetic. Therefore we are interested in Laplacian non-cospectral graphs with equal number of vertices, having equal Laplacian energies. Stevanović [24] has constructed Laplacian equienergetic threshold graphs. Fritscher et al. [10] discovered a family of Laplacian equienergetic unicyclic graphs. We now report some additional classes of such graphs.

The *line graph* of the graph G , denoted by $L(G)$, is a graph whose vertices corresponds to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G [14]. The k -th iterated line graph of G is defined

as $L^k(G) = L(L^{k-1}(G))$ where $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$. If G is a regular graph of order n_0 and of degree r_0 , then $L(G)$ is a regular graph of order $n_1 = n_0 r_0 / 2$ and of degree $r_1 = 2r_0 - 2$. Consequently, the order and degree of $L^k(G)$ are [3, 4]:

$$n_k = \frac{1}{2} n_{k-1} r_{k-1} \quad \text{and} \quad r_k = 2r_{k-1} - 2$$

where n_i and r_i stand for the order and degree of $L^i(G)$, $i = 0, 1, 2, \dots$. Therefore [3, 4],

$$(4.1) \quad n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2)$$

and

$$(4.2) \quad r_k = 2^k r_0 - 2^{k+1} + 2.$$

Theorem 4.1. [23] *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r , then the adjacency eigenvalues of $L(G)$ are*

$$\begin{aligned} \lambda_i + r - 2 \quad & i = 1, 2, \dots, n, \quad \text{and} \\ -2 \quad & n(r - 2)/2 \text{ times.} \end{aligned}$$

Theorem 4.2. [22] *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r , then the adjacency eigenvalues of \overline{G} , the complement of G , are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \dots, n$.*

Theorem 4.3. [16] *If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r , then its Laplacian eigenvalues are $r - \lambda_i$, $i = 1, 2, \dots, n$.*

For G being a regular graph of degree $r \geq 3$, and for $k \geq 2$, expressions for $\mathcal{E}(L^k(G))$ and $\mathcal{E}(\overline{L^k(G)})$ were reported in [20, 21].

Theorem 4.4. *If G is a regular graph of order n and of degree $r \geq 4$, then*

$$LE(L^2(G) \times K_2) = 4nr(r - 2).$$

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the adjacency eigenvalues of G . Then by Theorem 4.1, the adjacency eigenvalues of $L(G)$ are

$$(4.3) \quad \left. \begin{aligned} \lambda_i + r - 2 \quad & i = 1, 2, \dots, n, \quad \text{and} \\ -2 \quad & n(r - 2)/2 \text{ times.} \end{aligned} \right\}$$

Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, by Eq. (4.3) the adjacency eigenvalues of $L^2(G)$ are

$$(4.4) \quad \left. \begin{aligned} \lambda_i + 3r - 6 \quad & i = 1, 2, \dots, n, \quad \text{and} \\ 2r - 6 \quad & n(r - 2)/2 \text{ times,} \quad \text{and} \\ -2 \quad & nr(r - 2)/2 \text{ times.} \end{aligned} \right\}$$

Since $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and of degree $4r-2$, by Theorem 4.3 and Eq. (4.4), the Laplacian eigenvalues of $L^2(G)$ are

$$(4.5) \quad \left. \begin{array}{lll} r - \lambda_i & i = 1, 2, \dots, n, & \text{and} \\ 2r & n(r-2)/2 \text{ times,} & \text{and} \\ 4r - 4 & nr(r-2)/2 \text{ times.} & \end{array} \right\}$$

Using Theorem 3.1 and Eq. (4.5), the Laplacian eigenvalues of $L^2(G) \times K_2$ are

$$(4.6) \quad \left. \begin{array}{lll} r - \lambda_i & i = 1, 2, \dots, n, & \text{and} \\ 2r & n(r-2)/2 \text{ times,} & \text{and} \\ 4r - 4 & nr(r-2)/2 \text{ times,} & \text{and} \\ r - \lambda_i + 2 & i = 1, 2, \dots, n, & \text{and} \\ 2r + 2 & n(r-2)/2 \text{ times,} & \text{and} \\ 4r - 2 & nr(r-2)/2 \text{ times.} & \end{array} \right\}$$

The graph $L^2(G) \times K_2$ is a regular graph of order $nr(r-1)$ and of degree $4r-5$. By Eq. (4.6), the Laplacian energy of $L^2(G) \times K_2$ is computed as

$$(4.7) \quad \begin{aligned} LE(L^2(G) \times K_2) &= \sum_{i=1}^n |r - \lambda_i - (4r - 5)| + |2r - (4r - 5)| \frac{n(r-2)}{2} \\ &\quad + |4r - 4 - (4r - 5)| \frac{nr(r-2)}{2} + \sum_{i=1}^n |r - \lambda_i + 2 - (4r - 5)| \\ &\quad + |2r + 2 - (4r - 5)| \frac{n(r-2)}{2} + |4r - 2 - (4r - 5)| \frac{nr(r-2)}{2} \\ &= \sum_{i=1}^n |-\lambda_i - 3r + 5| + |-2r + 5| \frac{n(r-2)}{2} \\ &\quad + |1| \frac{nr(r-2)}{2} + \sum_{i=1}^n |-\lambda_i - 3r + 7| \\ &\quad + |-2r + 7| \frac{n(r-2)}{2} + |3| \frac{nr(r-2)}{2}. \end{aligned}$$

If d_{max} is the greatest vertex degree of a graph, then all its adjacency eigenvalues belongs to the interval $[-d_{max}, +d_{max}]$ [5]. In particular, the adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$, $i = 1, 2, \dots, n$.

If $r \geq 4$ then $\lambda_i + 3r - 5 > 0$, $\lambda_i + 3r - 7 > 0$, $2r - 5 > 0$, and $2r - 7 > 0$. Therefore by Eq. (4.7), and bearing in mind that $\sum_{i=1}^n \lambda_i = 0$,

$$\begin{aligned} LE(L^2(G) \times K_2) &= \sum_{i=1}^n \lambda_i + n(3r - 5) + (2r - 5) \frac{n(r - 2)}{2} + \frac{nr(r - 2)}{2} \\ &\quad + \sum_{i=1}^n \lambda_i + n(3r - 7) + (2r - 7) \frac{n(r - 2)}{2} + \frac{3nr(r - 2)}{2} \\ &= 4nr(r - 2). \end{aligned}$$

□

Corollary 4.1. *Let G be a regular graph of order n_0 and of degree $r_0 \geq 4$. Let n_k and r_k be the order and degree, respectively of the k -th iterated line graph $L^k(G)$ of G , $k \geq 2$. Then*

$$\begin{aligned} LE(L^k(G) \times K_2) &= 4n_{k-2}r_{k-2}(r_{k-2} - 2) = 4n_{k-1}(r_{k-1} - 2), \\ (4.8) \quad LE(L^k(G) \times K_2) &= 4n_0(r_0 - 2) \prod_{i=0}^{k-2} (2^i r_0 - 2^{i+1} + 2), \\ LE(L^k(G) \times K_2) &= 8(n_k - n_{k-1}) = 8n_k \left(\frac{r_k - 2}{r_k + 2} \right). \end{aligned}$$

From Eq. (4.8) we see that the energy of $L^k(G) \times K_2$, $k \geq 2$ is fully determined by the order n and degree $r \geq 4$ of G .

Theorem 4.5. *If G is a regular graph of order n and of degree $r \geq 3$, then*

$$LE(\overline{L^2(G)} \times K_2) = (nr - 4)(4r - 6) - 4.$$

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the adjacency eigenvalues of a regular graph G of order n and of degree $r \geq 3$. Then the adjacency eigenvalues of $L^2(G)$ are as given by Eq. (4.4).

Since $L^2(G)$ is a regular graph of order $nr(r - 1)/2$ and of degree $4r - 2$, by Theorem 4.2 and Eq. (4.4), the adjacency eigenvalues of $\overline{L^2(G)}$ are

$$(4.9) \quad \left. \begin{aligned} -\lambda_i - 3r + 5 & \quad i = 2, 3, \dots, n, & \quad \text{and} \\ -2r + 5 & \quad n(r - 2)/2 \text{ times,} & \quad \text{and} \\ 1 & \quad nr(r - 2)/2 \text{ times,} & \quad \text{and} \\ (nr(r - 1)/2) - 4r + 5. & & \end{aligned} \right\}$$

Since $\overline{L^2(G)}$ is a regular graph of order $nr(r-1)/2$ and of degree $(nr(r-1)/2)-4r+5$, by Theorem 4.3 and Eq. (4.9), the Laplacian eigenvalues of $\overline{L^2(G)}$ are

$$(4.10) \quad \left. \begin{array}{lll} (nr(r-1)/2) - r - \lambda_i & i = 2, 3, \dots, n, & \text{and} \\ (nr(r-1)/2) - 2r & n(r-2)/2 \text{ times,} & \text{and} \\ (nr(r-1)/2) - 4r + 4 & nr(r-2)/2 \text{ times,} & \text{and} \\ 0. & & \end{array} \right\}$$

Using Theorem 3.1 and Eq. (4.10), the Laplacian eigenvalues of $\overline{L^2(G)} \times K_2$ are

$$(4.11) \quad \left. \begin{array}{lll} (nr(r-1)/2) - r - \lambda_i & i = 2, 3, \dots, n, & \text{and} \\ (nr(r-1)/2) - 2r & n(r-2)/2 \text{ times,} & \text{and} \\ (nr(r-1)/2) - 4r + 4 & nr(r-2)/2 \text{ times,} & \text{and} \\ 0 & 1 \text{ time,} & \text{and} \\ (nr(r-1)/2) - r - \lambda_i + 2 & i = 2, 3, \dots, n, & \text{and} \\ (nr(r-1)/2) - 2r + 2 & n(r-2)/2 \text{ times,} & \text{and} \\ (nr(r-1)/2) - 4r + 6 & nr(r-2)/2 \text{ times,} & \text{and} \\ 2. & & \end{array} \right\}$$

The graph $\overline{L^2(G)} \times K_2$ is a regular graph of order $nr(r-1)$ and of degree $(nr(r-1)/2) - 4r + 6$. By Eq. (4.11),

$$\begin{aligned} LE(\overline{L^2(G)} \times K_2) &= \sum_{i=2}^n \left| \frac{nr(r-1)}{2} - r + \lambda_i - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ &+ \left| \frac{nr(r-1)}{2} - 2r - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \\ &+ \left| \frac{nr(r-1)}{2} - 4r + 4 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{nr(r-2)}{2} \\ &+ \left| 0 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ &+ \sum_{i=2}^n \left| \frac{nr(r-1)}{2} - r + \lambda_i + 2 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\ &+ \left| \frac{nr(r-1)}{2} - 2r + 2 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{n(r-2)}{2} \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{nr(r-1)}{2} - 4r + 6 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \frac{nr(r-2)}{2} \\
& + \left| 2 - \left(\frac{nr(r-1)}{2} - 4r + 6 \right) \right| \\
& = \sum_{i=2}^n |\lambda_i + 3r - 6| + |2r - 6| \frac{n(r-2)}{2} \\
& + \left| -2 \right| \frac{nr(r-2)}{2} + \left| \frac{-nr(r-1)}{2} + 4r - 6 \right| + \sum_{i=2}^n |\lambda_i + 3r - 4| \\
(4.12) \quad & + |2r - 4| \frac{n(r-2)}{2} + \left| 0 \right| \frac{nr(r-2)}{2} + \left| \frac{-nr(r-1)}{2} + 4r - 4 \right|.
\end{aligned}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$, $i = 1, 2, \dots, n$ [5]. Therefore if $r \geq 3$, then $\lambda_i + 3r - 6 \geq 0$, $\lambda_i + 3r - 4 \geq 0$, $2r - 6 \geq 0$, $2r - 4 \geq 0$, $(-nr(r-1)/2) + 4r - 6 < 0$, and $(-nr(r-1)/2) + 4r - 4 < 0$.

Then from Eq. (4.12), and bearing in mind that $\sum_{i=2}^n \lambda_i = -r$, we get

$$\begin{aligned}
LE \left(\overline{L^2(G)} \times K_2 \right) &= \sum_{i=2}^n \lambda_i + (n-1)(3r-6) + (r-3)n(r-2) + nr(r-2) \\
&+ \frac{nr(r-1)}{2} - 4r + 6 + \sum_{i=2}^n \lambda_i + (n-1)(3r-4) \\
&+ (r-2)n(r-2) + \frac{nr(r-1)}{2} - 4r + 4 \\
&= 2(nr-4)(2r-3) - 4.
\end{aligned}$$

□

Corollary 4.2. *Let G be a regular graph of order n_0 and of degree $r_0 \geq 3$. Let n_k and r_k be the order and degree, respectively of the k -th iterated line graph $L^k(G)$ of G , $k \geq 2$. Then*

$$\begin{aligned}
LE \left(\overline{L^k(G)} \times K_2 \right) &= (n_{k-2}r_{k-2} - 4)(4r_{k-2} - 6) - 4 \\
&= (2n_{k-1} - 4)(2r_{k-1} - 2) - 4, \\
(4.13) \quad LE \left(\overline{L^k(G)} \times K_2 \right) &= \left[\frac{n_0}{2^{k-2}} \prod_{i=0}^{k-2} (2^i r_0 - 2^{i+1} + 2) - 4 \right] (2^k r_0 - 2^{k+1} + 2) - 4,
\end{aligned}$$

$$LE\left(\overline{L^k(G)} \times K_2\right) = \frac{8n_k r_k}{2 + r_k} - 4(r_k + 1).$$

From Eq. (4.13) we see that the energy of $\overline{L^k(G)} \times K_2$, $k \geq 2$ is fully determined by the order n and degree $r \geq 3$ of G .

Theorem 4.6. *Let G_1 and G_2 be two Laplacian non-cospectral, regular graphs of the same order and of the same degree $r \geq 4$. Then for any $k \geq 2$, $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.*

Proof. If G is any graph with n vertices and m edges, then $G \times K_2$ has $2n$ vertices and $2m + n$ edges. Hence by repeated applications of Eqs. (4.1) and (4.2), $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ have same number of vertices and same number of edges. By Eqs. (4.5) and (4.6), if G_1 and G_2 are not Laplacian cospectral, then $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ are not Laplacian cospectral for all $k \geq 1$. Finally, Eq. (4.8) implies that $L^k(G_1) \times K_2$ and $L^k(G_2) \times K_2$ are Laplacian equienergetic. \square

Theorem 4.7. *Let G_1 and G_2 be two Laplacian non-cospectral, regular graphs of the same order and of the same degree $r \geq 3$. Then for any $k \geq 2$, $\overline{L^k(G_1)} \times K_2$ and $\overline{L^k(G_2)} \times K_2$ is a pair of Laplacian non-cospectral, Laplacian equienergetic graphs possessing same number of vertices and same number of edges.*

Proof. The proof is similar to that of Theorem 4.6 by using Eqs. (4.10), (4.11), and (4.13). \square

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