



Curvature Properties of Some Class of Minimal Hypersurfaces in Euclidean Spaces

Ryszard Deszcz^a, Małgorzata Głogowska^a, Miroslava Petrović-Torgašev^b, Leopold Verstraelen^c

^aDepartment of Mathematics, Wrocław University of Environmental and Life Sciences, Grunwaldzka 53, 50-357 Wrocław, POLAND

^bDepartment of Mathematics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, SERBIA

^cDepartement Wiskunde, Faculteit Wetenschappen, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Heverlee, BELGIUM

Dedicated to the birthday of Professor Mileva Provanović

Abstract. We determine curvature properties of pseudosymmetry type of some class of minimal 2-quasi-umbilical hypersurfaces in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geq 4$. We present examples of such hypersurfaces. The obtained results are used to determine curvature properties of biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$. Those hypersurfaces were recently investigated by Y. Fu in [38].

1. Introduction

Let M be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 4$, where $c = \frac{\tilde{\kappa}}{n(n+1)}$ and $\tilde{\kappa}$ are the sectional curvature and the scalar curvature of the ambient space, respectively. Let $\mathcal{U}_H \subset M$ be the set of all points at which the $(0, 2)$ -tensor H^2 is not expressed by a linear combination of the second fundamental tensor H and the metric tensor g of M . For precise definitions of the symbols used here, we refer to Section 2 of this paper (see also [19], [20] and [22]).

Curvature properties of pseudosymmetry type of hypersurfaces in semi-Riemannian spaces of constant curvature were investigated in several papers. In particular, hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, with the tensor H satisfying on $\mathcal{U}_H$

$$H^3 = \phi H^2 + \psi H + \rho g, \quad (1)$$

2010 *Mathematics Subject Classification.* 53B20, 53B30, 53B50; 53C25

Keywords. minimal hypersurface, 2-quasi-umbilical hypersurface, biharmonic hypersurface, Clifford torus, Clifford hypersurface, 2-quasi-Einstein manifold, manifold with pseudosymmetric Weyl tensor, Roter type manifold, Tachibana tensor.

Received: 24 August 2014; Accepted: 10 November 2014

Communicated by Ljubica Velimirović and Mića Stanković

The first named author is supported by a grant of the Technische Universität Berlin (Germany). The first named author is also supported by the Faculty of Science of the University of Kragujevac for the participation in the conference of XVIII Geometrical Seminar, May 18-21, 2014, Vrnjacka Banja, Serbia. The first and second named authors are supported by a grant of the Wrocław University of Environmental and Life Sciences (Poland). The third author is supported by the project 174012 of the Serbian Ministry of Education, Science and Technological Development. The third and the fourth named authors are supported partially by the Research Center of the Serbian Academy of Sciences and Arts (SANU) and the University of Kragujevac.

Email addresses: ryszard.deszcz@up.wroc.pl (Ryszard Deszcz), malgorzata.glogowska@up.wroc.pl (Małgorzata Głogowska), mirapt@kg.ac.rs (Miroslava Petrović-Torgašev), leopold.verstraelen@wis.kuleuven.be (Leopold Verstraelen)

for some functions ϕ, ψ and ρ , were investigated in the following papers: [9]–[13], [17]–[18], [21]–[23], [25], [28]–[31], [33], [36], [40], [48]–[52].

The main results of Section 3 are presented in Proposition 3.1 and Theorem 3.2. In Proposition 3.1 we present curvature properties of minimal hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1). In Theorem 3.2 we present curvature properties of minimal hypersurfaces M in semi-Euclidean spaces $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying (1) with $\rho = 0$, i.e.

$$H^3 = \phi H^2 + \psi H. \quad (2)$$

We also present examples of hypersurfaces satisfying (1), see Example 3.1(iii) and Example 3.2(ii).

In Section 4 we consider hypersurfaces M in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_H \subset M$ exactly three distinct principal curvatures λ_1, λ_2 and λ_3 such that

$$\lambda_1 = 0, \lambda_2 = -(n-2)\lambda, \lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda \neq 0, \quad (3)$$

where λ is a function on $\mathcal{U}_H$. Evidently, we have on $\mathcal{U}_H$: $\text{tr}(H) = 0$ and

$$H^3 = \phi H^2 + \psi H, \quad \phi = -(n-3)\lambda, \quad \psi = (n-2)\lambda^2, \quad \rho = 0. \quad (4)$$

In Proposition 4.1 we present curvature properties of hypersurfaces M in $N^{n+1}(c)$, $n \geq 4$, satisfying (3). Using results of that proposition we obtain curvature properties of hypersurfaces M in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geq 4$, satisfying (3). We also present examples of hypersurfaces satisfying (3), see Example 4.1 and Example 4.2(ii). We recall that a Riemannian manifold (M, g) , $n = \dim M$, isometrically immersed in an m -dimensional Euclidean space $\mathbb{E}^m$ is said to be *biharmonic submanifold* ([6]) if its mean curvature vector field $\vec{H}$ satisfies $\Delta \vec{H} = 0$, where Δ is the Laplace operator of M . For recent survey on biharmonic submanifolds we refer to the book of B.-Y. Chen [6]. It is clear that any minimal submanifold in $\mathbb{E}^m$ is trivially biharmonic. A biharmonic submanifold in $\mathbb{E}^m$ is called *proper biharmonic* if it is not minimal. Very recently, biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$ were investigated in [38]. In Theorem 3.2 of [38] it was stated that every biharmonic hypersurface M with three distinct principal curvatures in $\mathbb{E}^5$ is minimal. The principal curvatures: λ_1, λ_2 and λ_3 of M satisfy (3) with $n = 4$. In Theorem 4.3 we present curvature properties of those hypersurfaces.

2. Preliminaries

Throughout the paper all manifolds are assumed to be connected paracompact manifolds of class C^∞ . Let (M, g) be an n -dimensional, $n \geq 3$, semi-Riemannian manifold and let ∇ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on M .

We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$, respectively, by

$$\begin{aligned} (X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y, \\ \mathcal{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \end{aligned}$$

where A is a symmetric $(0, 2)$ -tensor on M and $X, Y, Z \in \Xi(M)$. The Ricci tensor S , the Ricci operator $\mathcal{S}$, the tensors S^2 and S^3 and the scalar curvature κ of (M, g) are defined by $S(X, Y) = \text{tr}\{Z \rightarrow \mathcal{R}(Z, X)Y\}$, $g(SX, Y) = S(X, Y)$, $S^2(X, Y) = S(SX, Y)$, $S^3(X, Y) = S^2(SX, Y)$ and $\kappa = \text{tr } \mathcal{S}$, respectively. The endomorphisms $C(X, Y)$ and $\text{conh}(\mathcal{R})(X, Y)$ are defined by

$$\begin{aligned} C(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g SY + SX \wedge_g Y - \frac{\kappa}{n-1}X \wedge_g Y)Z, \\ \text{conh}(\mathcal{R})(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g SY + SX \wedge_g Y), \end{aligned}$$

respectively. Now the $(0,4)$ -tensor G , the Riemann-Christoffel curvature tensor R , the Weyl conformal curvature tensor C and the conharmonic tensor $\text{conh}(R)$ of (M, g) are defined by

$$\begin{aligned} G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge_g X_2)X_3, X_4), \\ R(X_1, X_2, X_3, X_4) &= g(\mathcal{R}(X_1, X_2)X_3, X_4), \\ C(X_1, X_2, X_3, X_4) &= g(C(X_1, X_2)X_3, X_4), \\ \text{conh}(R)(X_1, X_2, X_3, X_4) &= g(\text{conh}(\mathcal{R})(X_1, X_2)X_3, X_4), \end{aligned}$$

respectively, where $X_1, X_2, \dots \in \Xi(M)$. We define the following subsets of M : $\mathcal{U}_R = \{x \in M \mid R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$, $\mathcal{U}_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ and $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$. We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$.

Let $\mathcal{B}$ be a tensor field sending any $X, Y \in \Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X, Y)$, and let B be a $(0,4)$ -tensor associated with $\mathcal{B}$ by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \quad (5)$$

The tensor B is said to be a *generalized curvature tensor* if the following conditions are satisfied

$$\begin{aligned} B(X_1, X_2, X_3, X_4) &= B(X_3, X_4, X_1, X_2), \\ B(X_1, X_2, X_3, X_4) + B(X_3, X_1, X_2, X_4) + B(X_2, X_3, X_1, X_4) &= 0. \end{aligned}$$

For $\mathcal{B}$ as above, let B be again defined by (5). We extend the endomorphism $\mathcal{B}(X, Y)$ to a derivation $\mathcal{B}(X, Y) \cdot$ of the algebra of tensor fields on M , assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$, for any smooth function f on M . For a $(0, k)$ -tensor field T , $k \geq 1$, we can define the $(0, k+2)$ -tensor $B \cdot T$ by

$$\begin{aligned} (B \cdot T)(X_1, \dots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \dots, X_k) \\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, \mathcal{B}(X, Y)X_k). \end{aligned}$$

In addition, if A is a symmetric $(0,2)$ -tensor then we define the $(0, k+2)$ -tensor $Q(A, T)$ by

$$\begin{aligned} Q(A, T)(X_1, \dots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \dots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \dots, X_k) - \dots - T(X_1, \dots, X_{k-1}, (X \wedge_A Y)X_k). \end{aligned}$$

The tensor $Q(A, T)$ is called the *Tachibana tensor of the tensors A and T* , or shortly the Tachibana tensor (see, e.g., [23]). We mention that in some papers the tensor $Q(g, R)$ is called the Tachibana tensor ([41], [42], [43], [47]).

For a symmetric $(0,2)$ -tensor E and a $(0, k)$ -tensor T , $k \geq 2$, we define their Kulkarni-Nomizu product $E \wedge T$ by ([18])

$$\begin{aligned} (E \wedge T)(X_1, X_2, X_3, X_4; Y_3, \dots, Y_k) \\ &= E(X_1, X_4)T(X_2, X_3, Y_3, \dots, Y_k) + E(X_2, X_3)T(X_1, X_4, Y_3, \dots, Y_k) \\ &\quad - E(X_1, X_3)T(X_2, X_4, Y_3, \dots, Y_k) - E(X_2, X_4)T(X_1, X_3, Y_3, \dots, Y_k). \end{aligned}$$

For instance, the following tensors are generalized curvature tensors: $R, C, G, \text{conh}(R)$ and $E \wedge F$, where E and F are symmetric $(0,2)$ -tensors. For a symmetric $(0,2)$ -tensor E we define the $(0,4)$ -tensor $\bar{E}$ by $\bar{E} = \frac{1}{2} E \wedge E$. In particular, we have $\bar{g} = G = \frac{1}{2} g \wedge g$ and

$$C = R - \frac{1}{n-2} g \wedge S + \frac{\kappa}{(n-2)(n-1)} G. \quad (6)$$

From (6) and the identity $Q(g, G) = 0$ we get immediately

$$Q(g, C) = Q(g, R - \frac{1}{n-2} g \wedge S) = Q(g, \text{conh}(R)). \quad (7)$$

We also have

Lemma 2.1. (cf. [27], Proposition 1) For any semi-Riemannian manifold (M, g) , $n \geq 4$, the following identities hold good

$$\begin{aligned} \operatorname{conh}(R) \cdot S &= C \cdot S - \frac{\kappa}{(n-2)(n-1)} Q(g, S), \\ R \cdot \operatorname{conh}(R) &= R \cdot C, \\ \operatorname{conh}(R) \cdot R &= C \cdot R - \frac{\kappa}{(n-2)(n-1)} Q(g, R), \\ \operatorname{conh}(R) \cdot \operatorname{conh}(R) &= C \cdot C - \frac{\kappa}{(n-2)(n-1)} Q(g, C). \end{aligned} \quad (8)$$

For a symmetric $(0, 2)$ -tensor A we define the endomorphism $\mathcal{A}$ and the tensors A^2 and A^3 by $g(\mathcal{A}X, Y) = A(X, Y)$, $A^2(X, Y) = A(\mathcal{A}X, Y)$ and $A^3(X, Y) = A^2(\mathcal{A}X, Y)$, respectively.

Lemma 2.2. Let E_1, E_2 and F be symmetric $(0, 2)$ -tensors at a point x of a semi-Riemannian manifold (M, g) , $n \geq 3$. (i) ([17], [18]) At x we have

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) + Q(F, E_1 \wedge E_2) = 0.$$

In particular, if $E = E_1 = E_2$ then at x we have

$$E \wedge Q(E, F) = -Q(F, \bar{E}).$$

Moreover (see, e.g., [21], Section 3)

$$Q(E, E \wedge F) = -Q(F, \bar{E}).$$

(ii) ([44], Lemma 3.2) At x we have

$$\begin{aligned} G \cdot F &= Q(g, F), \quad (g \wedge F) \cdot F = Q(g, F^2), \\ -(g \wedge F) \cdot (g \wedge F) &= Q(F^2, G). \end{aligned}$$

Moreover, if A is a symmetric $(0, 2)$ -tensor and B a generalized curvature tensor then

$$G \cdot A = Q(g, A), \quad G \cdot B = Q(g, B).$$

(iii) (see, e.g., [37], Lemma 2.4 (iii)) At x we have

$$Q(E_1, E_2 \wedge F) + Q(E_2, F \wedge E_1) + Q(F, E_1 \wedge E_2) = 0.$$

As an immediate consequence of (6) and Lemma 2.2(ii) we get (also see [28], p. 217)

Lemma 2.3. On any semi-Riemannian manifold (M, g) , $n \geq 4$, we have the following identity

$$C \cdot S = R \cdot S - \frac{1}{n-2} Q(g, S^2 - \frac{\kappa}{n-1} S). \quad (9)$$

Let B_{hijk} , T_{hijk} , and A_{ij} be the local components of generalized curvature tensors B and T and a symmetric $(0, 2)$ -tensor A on M , respectively, where $h, i, j, k, l, m, p, q \in \{1, 2, \dots, n\}$. The local components $(B \cdot T)_{hijklm}$ and $Q(A, T)_{hijklm}$ of the tensors $B \cdot T$, $Q(A, T)$, $B \cdot A$ and $Q(g, A)$ are the following

$$\begin{aligned} (B \cdot T)_{hijklm} &= g^{pq}(T_{pijk}B_{qhl m} + T_{hpjk}B_{qilm} + T_{hipk}B_{qjlm} + T_{hijp}B_{qklm}), \\ Q(A, T)_{hijklm} &= A_{hl}T_{mijk} + A_{il}T_{hmjk} + A_{jl}T_{himk} + A_{kl}T_{hijm} \\ &\quad - A_{lm}T_{lijk} - A_{im}T_{hljk} - A_{jm}T_{hil k} - A_{km}T_{hijl}, \\ (B \cdot A)_{hk lm} &= g^{pq}(A_{pk}B_{qhl m} + A_{ph}B_{qkl m}), \\ Q(g, A)_{hk lm} &= g_{hl}A_{km} + g_{kl}A_{hm} - g_{lm}A_{kl} - g_{km}A_{hl}. \end{aligned}$$

The manifold (M, g) , $n \geq 3$, is said to be an *Einstein manifold* [1] if $S = \frac{\kappa}{n} g$ on M .

Einstein manifolds form a subclass of the class of quasi-Einstein manifolds. The semi-Riemannian manifold (M, g) , $n \geq 3$, is called a *quasi-Einstein manifold* if $\text{rank}(S - \alpha g) = 1$ on $\mathcal{U}_S$, where α is some function on this set. Every warped product manifold $\bar{M} \times_F \tilde{N}$ of an 1-dimensional $(\bar{M}, \bar{g})$ base manifold and an 2-dimensional manifold $(\tilde{N}, \tilde{g})$ or an $(n - 1)$ -dimensional Einstein manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, with a warping function F , is a quasi-Einstein manifold. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [17], [21], [25], [31] and [40], see also [20]. We refer to [8] and [27] for recent results on quasi-Einstein manifolds.

The semi-Riemannian manifold (M, g) , $n \geq 3$, is called a *2-quasi-Einstein manifold* if $\text{rank}(S - \alpha g) \leq 2$ on $\mathcal{U}_S$ and $\text{rank}(S - \alpha g) = 2$ on some open non-empty subset of $\mathcal{U}_S$, where α is some function on $\mathcal{U}_S$. It is clear that every warped product manifold $\bar{M} \times_F \tilde{N}$ of an 2-dimensional $(\bar{M}, \bar{g})$ base manifold and an 2-dimensional manifold $(\tilde{N}, \tilde{g})$ or an $(n - 2)$ -dimensional Einstein manifold $(\tilde{N}, \tilde{g})$, $n \geq 5$, with a warping function F , is a 2-quasi-Einstein manifold. Therefore some exact solutions of the Einstein field equations are 2-quasi-Einstein manifolds, e.g. the Reissner-Nordström-de Sitter type spacetimes are such manifolds (see, e.g., [44]). It seems that the Reissner-Nordström spacetime is the "oldest" example of a 2-quasi-Einstein warped product manifold. It is easy to see that every 2-quasi-umbilical hypersurface in a space of constant curvature is a 2-quasi-Einstein manifold (see Remark 3.1). We refer to [24] for recent results on 2-quasi-Einstein warped product manifolds.

3. Hypersurfaces in spaces of constant curvature

Let M be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, \tilde{g})$ of dimension $n + 1$, $n \geq 3$. Let g be the metric tensor induced on M from $\tilde{g}$. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections corresponding to the metric tensors g and $\tilde{g}$, respectively. We denote by ξ a local unit normal vector field on M in N and let $\varepsilon = \tilde{g}(\xi, \xi) = \pm 1$. We can write the *Gauss formula* and the *Weingarten formula* of (M, g) in $(N, \tilde{g})$ in the form: $\tilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y) \xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where X, Y are vector fields tangent to M . H is the *second fundamental tensor* and $\mathcal{A}$ the *shape operator* of (M, g) in $(N, \tilde{g})$. We have $H(X, Y) = g(\mathcal{A}X, Y)$, for any vectors fields X, Y tangent to M . Further, we set $H^p(X, Y) = g(\mathcal{A}^p X, Y)$, $p = 1, 2, \dots$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by H_{hk}^p the local components of the tensor H^p .

According to [4], [5], [7], [46], [53], a hypersurface M in an $(n + 1)$ -dimensional Riemannian manifold N is said to be *quasi-umbilical*, resp., *2-quasi-umbilical*, at a point $x \in M$ if it has a principal curvature with multiplicity $n - 1$, resp., $n - 2$, i.e. when the principal curvatures of M at x are given by $\lambda_1, \lambda_2 = \lambda_3 = \dots = \lambda_n$, resp., $\lambda_1, \lambda_2, \lambda_3 = \lambda_4 = \dots = \lambda_n$. If M is a hypersurface in an $(n + 1)$ -dimensional semi-Riemannian manifold N then M is called *quasi-umbilical* (see, e.g., [34], [40]), resp., *2-quasi-umbilical* (see, e.g., [36], [40]), at a point $x \in M$ when $\text{rank}(H - \alpha g) = 1$, resp., $\text{rank}(H - \alpha g) = 2$, holds at x , for some $\alpha \in \mathbb{R}$.

We recall that a hypersurface M in a semi-Riemannian conformally flat manifold N is quasi-umbilical at a point $x \in M$ if and only if its Weyl conformal curvature tensor C vanishes at this point ([34], Theorem 4.1). Thus a point $x \in M$ is a non-quasi-umbilical point of M if and only if the tensor C is non-zero at x , i.e. $x \in \mathcal{U}_C \subset M$.

We denote by R and $\tilde{R}$ the Riemann-Christoffel curvature tensors of (M, g) and $(N, \tilde{g})$, respectively. Let $x^r = x^r(y^k)$ be the local parametric expression of (M, g) in $(N, \tilde{g})$, where y^k and x^r are local coordinates of M and N , respectively, $h, i, j, k \in \{1, 2, \dots, n\}$ and $p, r, t, u \in \{1, 2, \dots, n + 1\}$. The *Gauss equation* of (M, g) in $(N, \tilde{g})$ reads

$$R_{hijk} = \tilde{R}_{prt u} B_h^p B_i^r B_j^t B_k^u + \varepsilon (H_{hk} H_{ij} - H_{hj} H_{ik}), \quad B_k^r = \frac{\partial x^r}{\partial y^k}, \quad (10)$$

where $\tilde{R}_{prt u}$, R_{hijk} and H_{hk} are the local components of the tensors $\tilde{R}$, R and H , respectively.

Let M be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n + 1 - s)$, $n \geq 4$, where $c = \frac{\tilde{\kappa}}{n(n+1)}$ and $\tilde{\kappa}$ are the sectional curvature and the scalar

curvature of the ambient space, respectively. Now (10) turns into

$$R_{hijk} = \varepsilon (H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\widetilde{\kappa}}{n(n+1)} G_{hijk}, \quad \varepsilon = \pm 1. \quad (11)$$

Contracting (11) with g^{ij} and g^{kh} we obtain

$$S_{hk} = \varepsilon (tr(H)H_{hk} - H_{hk}^2) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)} g_{hk}, \quad (12)$$

$$\kappa = \varepsilon ((tr(H))^2 - tr(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}, \quad (13)$$

respectively, where $tr(H^2) = g^{hk}H_{hk}^2$ and S_{hk} are the local components of the Ricci tensor S of M . It is known that on M we have ([34])

$$R \cdot R - Q(S, R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, C). \quad (14)$$

In particular, if the ambient space is a semi-Euclidean space $\mathbb{E}_s^{n+1}$ then (14) reduces to

$$R \cdot R = Q(S, R). \quad (15)$$

Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (1) on $\mathcal{U}_H$. We define on $\mathcal{U}_H$ the following functions ([48], eq. (34)):

$$\begin{aligned} \beta_1 &= \varepsilon (\phi - tr(H)), \\ \beta_2 &= -\frac{\varepsilon}{n-2} (\phi (2tr(H) - \phi) - (tr(H))^2 - \psi - (n-2)\varepsilon\mu), \\ \beta_3 &= \varepsilon\mu tr(H) + \frac{1}{n-2} (\psi (2tr(H) - \phi) + (n-3)\rho), \\ \beta_4 &= \beta_3 - \varepsilon\beta_2 tr(H) + \frac{(n-1)\widetilde{\kappa}\beta_1}{n(n+1)}, \end{aligned} \quad (16)$$

where the functions: ϕ, ψ, ρ, μ are defined by (1) and

$$\mu = \frac{1}{n-2} \left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1} \right), \quad (17)$$

respectively. We also have on $\mathcal{U}_H$ ([48], eqs. (43), (52), (45), (46)):

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S) + \rho Q(g, H) - \varepsilon\beta_1 Q(H, H^2), \quad (18)$$

$$C \cdot S = \beta_1 Q(H, S) + \beta_2 Q(g, S) + \beta_4 Q(g, H), \quad (19)$$

$$\begin{aligned} (n-2)R \cdot C &= (n-2)Q(S, R) \\ &\quad - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)} Q(g, R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)} Q(S, G) \\ &\quad + \rho Q(H, G) + (\phi - tr(H))g \wedge Q(H, H^2), \end{aligned} \quad (20)$$

$$\begin{aligned} (n-2)C \cdot R &= (n-3)Q(S, R) \\ &\quad + \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2 - 3n + 3)\widetilde{\kappa}}{n(n+1)} \right) Q(g, R) \\ &\quad - \frac{(n-3)\widetilde{\kappa}}{n(n+1)} Q(S, G) + (\phi - tr(H))H \wedge Q(g, H^2), \end{aligned} \quad (21)$$

where $\beta_1, \dots, \beta_4$ are defined by (16).

- Example 3.1.** (i) (Example 1.1, [54]) The Clifford hypersurfaces in $N^n(c)$, $c \neq 0$, $n \geq 4$. (a) For $c > 0$ we set $N^n(c) = S^n(c) = \{x \in \mathbb{R}^{n+1} : \langle x, x \rangle = \frac{1}{c}\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{n+1}$. For $1 \leq m \leq n-2$, $t \in (0, \frac{\pi}{2})$, let $M_{m,n-m-1}(c, t) = S^m(\frac{c}{\sin^2 t}) \times S^{n-m-1}(\frac{c}{\cos^2 t})$. We view $x = (x_1, x_2) \in M_{m,n-m-1}(c, t)$ as a vector in $\mathbb{R}^{n+1} = \mathbb{R}^{m+1} \times \mathbb{R}^{n-m}$, then $x \in S^n(c)$. This is the standard isometric embedding of $M_{m,n-m-1}(c, t)$ into $S^n(c)$. In this situation, for suitably chosen unit normal vector field, $M_{m,n-m-1}(c, t)$ has two distinct principal curvatures $\rho_1 = \sqrt{c} \cot t$ of the multiplicity m and $\rho_2 = -\sqrt{c} \tan t$ of the multiplicity $n-m-1$. (b) For $c < 0$ we set $N^n(c) = H^n(c) = \{x \in \mathbb{R}_1^{n+1} : \langle x, x \rangle_1 = \frac{1}{c}, x^{n+1} > 0\}$. Here $\langle x, y \rangle_1 = x^1 y^1 + \cdots + x^n y^n - x^{n+1} y^{n+1}$ is the standard Lorentzian inner product on $\mathbb{R}_1^{n+1}$. For $1 \leq m \leq n-2$, $t \in (0, +\infty)$, let $M_{m,n-m-1}(c, t) = S^m(\frac{-c}{\sinh^2 t}) \times H^{n-m-1}(\frac{-c}{\cosh^2 t})$. Then $M_{m,n-m-1}(c, t)$ is an embedded hypersurface in $H^n(c)$, and for suitably chosen unit normal vector field, it has two distinct principal curvatures $\rho_1 = -c \coth t$ of the multiplicity m and $\rho_2 = -c \tanh t$ of the multiplicity $n-m-1$.
- (ii) (a) If $2 \leq m \leq n-3$ and $(m-1)c_1 \neq (n-m-2)c_2$, where $c_1 = \frac{c}{\sin^2 t}$, $c_2 = \frac{c}{\cos^2 t}$, $t \in (0, \frac{\pi}{2})$ then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor R of $M_{m,n-m-1}(c, t)$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, i.e. $M_{m,n-m-1}(c, t)$ is a Roter type hypersurface. (b) If $2 \leq m \leq n-3$ and $(m-1)c_1 \neq (n-m-2)c_2$, where $c_1 = \frac{-c}{\sinh^2 t}$, $c_2 = \frac{-c}{\cosh^2 t}$, $t \in (0, +\infty)$, then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor R of $M_{m,n-m-1}(c, t)$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, i.e. $M_{m,n-m-1}(c, t)$ is a Roter type hypersurface. (c) The Roter type manifolds (and in particular, hypersurfaces in space forms) were studied among others in the papers: [19], [20], [22], [25], [26], [32], [39] and [44].
- (iii) Let M be a n -dimensional hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let M the cone over the Clifford hypersurface $M_{m,n-m-1}(c, t)$ defined in (i). We refer to Section 3 of [45] for precise definition and properties of cones. In particular, from Section 3 of [45] it follows immediately that M has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, $t \in \mathbb{R}^+$, of the multiplicity 1, m and $n-m-1$, respectively. Thus we see that the cone over the Clifford hypersurface $M_{m,n-m-1}(c, t)$, presented in (i) is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures and satisfying at every point $\mathcal{U}_H = M$ the equation (1) with $\rho = 0$, i.e. (2).
- (iv) We mention that an example of a hypersurface M in $\mathbb{E}^{n+1}$, $n \geq 4$, satisfying (1) on $\mathcal{U}_H \subset M$, with non-zero function ρ and $\phi = \text{tr}(H)$, is presented in [52].
- (v) The Cartan hypersurfaces of dimension 6, 12 or 24 satisfy (2), with $\phi = \text{tr}(H) = 0$. Curvature properties of these hypersurfaces are presented in [18] (Theorem 4.3).

Proposition 3.1. If M is a minimal hypersurface in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, $n \geq 4$,

satisfying (1) on $\mathcal{U}_H \subset M$ then the following conditions are satisfied on this set: (14) and

$$\begin{aligned}\beta_1 &= \varepsilon \phi, \quad \beta_2 = \frac{\varepsilon}{n-2}(\phi^2 + \psi + (n-2)\varepsilon\mu), \\ \beta_3 &= \frac{1}{n-2}((n-3)\rho - \psi\phi), \quad \beta_4 = \beta_3 + \frac{(n-1)\widetilde{\kappa}\varepsilon\phi}{n(n+1)},\end{aligned}\quad (22)$$

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S) + \rho Q(g, H) - \phi Q(H, H^2), \quad (23)$$

$$C \cdot S = \varepsilon \phi Q(H, S) + \beta_2 Q(g, S) + \beta_4 Q(g, H), \quad (24)$$

$$\begin{aligned}(n-2)R \cdot C &= (n-2)Q(S, R) \\ &\quad - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)} Q(g, R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)} Q(S, G) \\ &\quad + \rho Q(H, G) + \phi g \wedge Q(H, H^2),\end{aligned}\quad (25)$$

$$\begin{aligned}(n-2)C \cdot R &= (n-3)Q(S, R) \\ &\quad + \left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)} \right) Q(g, R) \\ &\quad - \frac{(n-3)\widetilde{\kappa}}{n(n+1)} Q(S, G) + \phi H \wedge Q(g, H^2),\end{aligned}\quad (26)$$

$$(\phi\psi + \rho)H = A^2 + \varepsilon(\phi^2 + \psi)A - \phi\rho g, \quad (27)$$

$$A^3 = -\varepsilon(\phi^2 + 2\psi)A^2 + (2\phi\rho - \psi^2)A - \varepsilon\rho^2 g, \quad (28)$$

$$\begin{aligned}(\phi\psi + \rho)^2 R &= \frac{\varepsilon}{2} (A^2 + \varepsilon(\phi^2 + \psi)A - \phi\rho g) \wedge (A^2 + \varepsilon(\phi^2 + \psi)A - \phi\rho g) \\ &\quad + \frac{(\phi\psi + \rho)^2\widetilde{\kappa}}{n(n+1)} G,\end{aligned}\quad (29)$$

where $\beta_1, \dots, \beta_4$ are defined by (22) and

$$A = S - \frac{(n-1)\widetilde{\kappa}}{n(n+1)} g. \quad (30)$$

Proof. Since M is a minimal hypersurface, (16) and (18)-(21) turn into (22)-(26), respectively. From (1), (12) and (30) we get easily

$$A = -\varepsilon H^2, \quad A^2 = H^4, \quad A^3 = -\varepsilon H^6, \quad (31)$$

$$H^4 = (\phi^2 + \psi)H^2 + (\phi\psi + \rho)H + \phi\rho g, \quad (32)$$

$$H^6 = (\phi^2 + \psi)H^4 + \phi(\phi\psi + 2\rho)H^2 + \psi(\phi\psi + \rho)H + \rho(\phi\psi + \rho)g. \quad (33)$$

Now (27)-(29) are immediate consequences of (11) and (31)-(33). Our proposition is thus proved.

Remark 3.1. (i) Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures λ_1, λ_2 and λ_3 , then (18)-(21) hold on $\mathcal{U}_H$ with $\varepsilon = 1$ and

$$\phi = \lambda_1 + \lambda_2 + \lambda_3, \quad \psi = -\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3, \quad \rho = \lambda_1\lambda_2\lambda_3. \quad (34)$$

(ii) Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures λ_1, λ_2 and $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda$, then from (12) it follows that

$$\text{rank} \left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} + \lambda (\text{tr}(H) - \lambda) \right) g \right) = 2 \quad (35)$$

on $\mathcal{U}_H$. Moreover, the following condition holds on $\mathcal{U}_H$ (see [36], p. 53)

$$C \cdot C = -\frac{(n-3)\lambda_1\lambda_2}{(n-1)(n-2)} Q(g, C). \quad (36)$$

We refer to [13], [35], [19], [27] and [32] for results on semi-Riemannian manifolds (M, g) , $\dim M \geq 4$, and in particular, on hypersurfaces M in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_C \subset M$

$$C \cdot C = L Q(g, C), \quad (37)$$

where L is some function on this set. We mention that the warped product manifold $\overline{M} \times_F \widetilde{N}$, of manifolds $(\overline{M}, \overline{g})$, $\dim \overline{M} = 2$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = 2$, and the warping function F satisfies (37) on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ (see, e.g., [20]). We also mention that the warped product manifold $\overline{M} \times_F \widetilde{N}$, of manifolds $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = 3$, and the warping function F satisfies on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$

$$R \cdot R - Q(S, R) = L Q(g, C),$$

where L is some function on this set ([11]).

Proposition 3.1 leads to the following

Theorem 3.2. *If M is a minimal hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying (2) on $\mathcal{U}_H \subset M$ then the following conditions are satisfied on this set: (15) and*

$$\begin{aligned} \phi\psi H &= S^2 + \varepsilon(\phi^2 + \psi)S, \\ S^3 &= -\varepsilon(\phi^2 + 2\psi)S^2 - \psi^2 S, \\ (\phi\psi)^2 R &= \frac{\varepsilon}{2}(S^2 + \varepsilon(\phi^2 + \psi)S) \wedge (S^2 + \varepsilon(\phi^2 + \psi)S), \\ R \cdot S &= \varepsilon\phi Q(H, S), \\ C \cdot S &= \varepsilon\phi Q(H, S) - \frac{\psi\phi}{n-2} Q(g, H) + \frac{\varepsilon}{n-2}(\phi^2 + \psi + \frac{\varepsilon\kappa}{n-1}) Q(g, S), \\ (n-2)R \cdot C &= (n-2)Q(S, R) - \varepsilon\phi g \wedge Q(H, S), \\ (n-2)C \cdot R &= (n-3)Q(S, R) + (\varepsilon\psi + \frac{\kappa}{n-1}) Q(g, R) - \varepsilon\phi H \wedge Q(g, S). \end{aligned}$$

Example 3.2. (i) Let $\mathcal{M}$ be a $(n-1)$ -dimensional hypersurface in n -dimensional standard unit sphere $S^n(1)$ in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let $\mathcal{M}$ be the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$, $c_1 = r_1^{-1}$, $c_2 = r_2^{-1}$, $r_1 = \sqrt{\frac{p}{n-1}}$, $r_2 = \sqrt{\frac{n-p-1}{n-1}}$, $1 \leq p \leq n-2$. It is well-known that $\mathcal{M}$ is a minimal hypersurface of $S^n(1)$ having at every point exactly two principal curvatures ρ_1 and ρ_2 of the multiplicity p and $n-p-1$, respectively, satisfying

$$\rho_1\rho_2 + 1 = 0, \quad \rho_i^2 = r_i^{-2} - 1, \quad i = 1, 2. \quad (38)$$

(ii) Let M be a n -dimensional hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let M be the cone over $\mathcal{M}$. We refer to Section 3 of [45] for precise definition and properties of such hypersurfaces. In particular, from Section 3 of [45] it follows immediately that M has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, $t \in \mathbb{R}^+$, of the multiplicity 1, p and $n-p-1$, respectively. Thus we see that the cone M over the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$ is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures satisfying at every point (2). Using (38) we can check that $\psi = -\lambda_2\lambda_3 = t^{-2}$ and

$$\begin{aligned} \phi^2 &= (\lambda_2 + \lambda_3)^2 = \frac{1}{t^2}(\rho_1 + \rho_2)^2 = \frac{1}{t^2}(\rho_1^2 + \rho_2^2 - 2) = \frac{1}{t^2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - 4\right) \\ &= \frac{1}{t^2}\left(\frac{(n-1)^2}{p(n-p-1)} - 4\right) = \frac{((n-p-1)+p)^2 - 4p(n-p-1)}{p(n-p-1)t^2} = \frac{(n-2p-1)^2}{p(n-p-1)t^2}. \end{aligned}$$

If $p \neq n - p - 1$ then in view of Theorem 3.2 the Riemann-Christoffel curvature tensor R of the cone M is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. We refer to [50] and [52] for further results on hypersurfaces with the curvature tensor having the above presented property.

Remark 3.2. (i) Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and let the condition

$$H^3 = \operatorname{tr}(H)H^2 + \psi H + \rho g,$$

be satisfied on $\mathcal{U}_H \subset M$, where ψ and ρ are some functions on this set. Using the identity (9), and (3.6) and (3.7) of [23] we get on $\mathcal{U}_H$

$$C \cdot S = \left(\varepsilon\psi + \frac{\kappa}{(n-2)(n-1)} - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)} \right) Q(g, S) + \frac{n-3}{n-2} Q(g, S^2). \quad (39)$$

(ii) (cf., [29], Lemma 4.2) Let M is a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ the relation

$$H^3 = \operatorname{tr}(H)H^2 - \frac{\varepsilon\kappa}{n-1} H.$$

Now (39), by (3.9) of [23], and the conditions $\widetilde{\kappa} = 0$ and $\psi = -\frac{\varepsilon\kappa}{n-1}$, reduces on $\mathcal{U}_H$ to

$$C \cdot S = 0. \quad (40)$$

Hypersurfaces satisfying (40) were investigated among others in: [2], [9]–[12], [21], [28]–[30].

(iii) Let (M, g) , $n \geq 4$, be a non-Riemannian semi-Riemannian manifolds with parallel Weyl tensor ($\nabla C = 0$), which are in addition non-locally symmetric ($\nabla R \neq 0$) and non-conformally flat ($C \neq 0$). Such manifolds are called essentially conformally symmetric manifolds, e.c.s. manifolds, in short (see, e.g., [14]). Certain e.c.s. metrics are realized on compact manifolds ([15], [16]). As it was stated in [14], e.c.s. manifolds are semisymmetric manifolds ($R \cdot R = 0$) satisfying: $\kappa = 0$, $S^2 = 0$ and $C(\widetilde{S}X_1, X_2, X_3, X_4) = 0$, for any $X_1, \dots, X_4 \in \Xi(M)$. Thus, in view of Lemma 2.3, we see that (40) holds on every e.c.s. manifold.

4. Some special minimal 2-quasi-umbilical hypersurfaces

In this section we consider hypersurfaces M in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_H \subset M$ exactly three distinct principal curvatures λ_1 , λ_2 and $\lambda_3 = \lambda$ such that (3) is satisfied. Thus at every point of $\mathcal{U}_H$ we have: $\lambda \neq 0$, $\operatorname{tr}(H) = 0$ and

$$\operatorname{rank}(H - \lambda g) = 2, \quad (41)$$

$$\operatorname{rank}\left(S - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \lambda^2\right)g\right) = 2. \quad (42)$$

The last condition follows immediately from (35). Therefore $\mathcal{U}_H$ is a minimal, 2-quasi-umbilical and 2-quasi Einstein open submanifold of M . Evidently, (36) reduces to

$$C \cdot C = 0. \quad (43)$$

This, together with (7) and (8), yields

$$\operatorname{conh}(R) \cdot \operatorname{conh}(R) = -\frac{\kappa}{(n-2)(n-1)} Q(g, \operatorname{conh}(R)). \quad (44)$$

Furthermore (1), (12), (13), (16) and (18)-(34) give (4) and

$$S = -H^2 + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}g, \quad \kappa = -tr(H^2) + \frac{(n-1)\widetilde{\kappa}}{n+1}, \quad (45)$$

$$\begin{aligned} \beta_1 &= \phi, \quad \beta_2 = \frac{1}{n-2}(\phi^2 + \psi + (n-2)\mu), \\ \beta_3 &= -\frac{\psi\phi}{n-2}, \quad \beta_4 = \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\psi}{n-2}\right)\phi, \end{aligned} \quad (46)$$

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)}Q(g, S) - \phi Q(H, H^2), \quad (47)$$

$$\begin{aligned} C \cdot S &= \phi Q(H, S) + \frac{1}{n-2}(\phi^2 + \psi + (n-2)\mu)Q(g, S) \\ &\quad + \left(\frac{(n-1)\widetilde{\kappa}\phi}{n(n+1)} - \frac{1}{n-2}\psi\phi\right)Q(g, H), \end{aligned} \quad (48)$$

$$\begin{aligned} (n-2)R \cdot C &= (n-2)Q(S, R) + \phi g \wedge Q(H, H^2) \\ &\quad - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}Q(g, R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}Q(S, G), \end{aligned} \quad (49)$$

$$\begin{aligned} (n-2)C \cdot R &= (n-3)Q(S, R) \\ &\quad + \left(\frac{\kappa}{n-1} + \psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right)Q(g, R) \\ &\quad - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}Q(S, G) + \phi H \wedge Q(g, H^2). \end{aligned} \quad (50)$$

Next, using (4) and (12), we find

$$H^2 = -S + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}g, \quad (51)$$

$$H^4 = (\phi^2 + \psi)H^2 + \phi\psi H, \quad (52)$$

$$\begin{aligned} \phi\psi H &= S^2 - \left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi\right)S \\ &\quad + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)g. \end{aligned} \quad (53)$$

Further, (28) turns into

$$\begin{aligned} S^3 &= \left(\frac{3(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - 2\psi\right)S^2 + \left(\psi\left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \psi\right) - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)^2\right)S \\ &\quad + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) - \psi\left(2\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right)g. \end{aligned} \quad (54)$$

We note that by the Gauss equation (11) and (53) we obtain on $\mathcal{U}_H$ the following relation

$$\begin{aligned} &2(\phi\psi)^2\left(R - \frac{\widetilde{\kappa}}{n(n+1)}G\right) \\ &= \left(S^2 - \left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi\right)S\right) \wedge \left(S^2 - \left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi\right)S\right). \end{aligned} \quad (55)$$

It is obvious that if the hypersurface M in $N^{n+1}(c)$, $n \geq 4$, has at every point exactly three distinct principal curvatures then $M = \mathcal{U}_H$. In this case we also have $M = \mathcal{U}_S = \mathcal{U}_C$.

The above presented results lead immediately to the following proposition.

Proposition 4.1. Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having exactly three distinct principal curvatures λ_1, λ_2 and λ_3 satisfying at every point of M : $\lambda_1 = 0, \lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda \neq 0$. Then M is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (14) and (41)–(55).

From the last proposition, (14) and (17) we immediately get the following.

Proposition 4.2. Let M be a hypersurface in an Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures λ_1, λ_2 and λ_3 satisfying at every point of M : $\lambda_1 = 0, \lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda \neq 0$. Then M is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (15), (43), (44) and

$$\begin{aligned} S &= H^2, \quad \kappa = -\text{tr}(H^2) = -(n-2)(n-1)\lambda^2, \\ \phi\psi H &= S^2 + (\phi^2 + \psi)S, \\ S^3 &= -(\phi^2 + 2\psi)S^2 - \psi^2S, \\ \phi &= -(n-3)\lambda, \quad \psi = (n-2)\lambda^2, \quad \mu = \frac{\kappa}{(n-2)(n-1)}, \\ \text{rank} \left(S - \frac{\kappa}{(n-2)(n-1)}g \right) &= 2, \\ R &= \frac{1}{2(\phi\psi)^2} (S^2 + (\phi^2 + \psi)S) \wedge (S^2 + (\phi^2 + \psi)S), \\ R \cdot S &= \phi Q(H, S) = \frac{n-1}{\kappa} Q(S, S^2), \\ C \cdot S &= \phi Q(H, S) + \frac{\phi^2}{n-2} Q(g, S) - \frac{\phi\psi}{n-2} Q(g, H) \\ &= \frac{n-1}{\kappa} Q\left(S - \frac{\kappa}{(n-2)(n-1)}g, S^2 - \frac{\kappa}{n-1}S\right), \\ (n-2)R \cdot C &= (n-2)Q(S, R) - \phi g \wedge Q(H, S), \\ (n-2)C \cdot R &= (n-3)Q(S, R) - \phi H \wedge Q(g, S). \end{aligned}$$

Theorem 4.3. Let M be a hypersurface in an Euclidean space $\mathbb{E}^5$ having exactly three distinct principal curvatures λ_1, λ_2 and λ_3 satisfying at every point of M : $\lambda_1 = 0, \lambda_2 = -2\lambda$ and $\lambda_3 = \lambda_4 = \lambda \neq 0$. Then M is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying: (15), (43), (44) and

$$\begin{aligned} \lambda^2 &= -\frac{\kappa}{6}, \quad \lambda H = \frac{3}{\kappa}S^2 - \frac{3}{2}S, \quad H^2 = -S, \\ S^3 &= \frac{5\kappa}{6}S^2 - \frac{\kappa}{9}S, \quad \text{rank} \left(S - \frac{\kappa}{6}g \right) = 2, \\ R &= -\frac{27}{\kappa^3} \left(S^2 - \frac{\kappa}{2}S \right) \wedge \left(S^2 - \frac{\kappa}{2}S \right), \\ R \cdot S &= \frac{3}{\kappa} Q(S, S^2), \\ C \cdot S &= \frac{3}{\kappa} Q\left(S - \frac{\kappa}{6}g, S^2 - \frac{\kappa}{3}S\right), \\ R \cdot C &= Q(S, R) + \frac{3}{2\kappa} g \wedge Q(S^2, S) \\ C \cdot R &= \frac{1}{2} Q(S, R) + \frac{3}{2\kappa} S^2 \wedge Q(g, S) - \frac{3}{8} Q(g, S \wedge S). \end{aligned}$$

Example 4.1. If $p = 1$ then the hypersurface M defined in Example 3.2 (ii) has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, of multiplicity 1, 1 and $n - 2$, respectively. Further, we set $\lambda = \lambda_3 = \frac{1}{t}\rho_2 = \frac{1}{\sqrt{n-2}t}$. This by (38) yields $\lambda_2 = -(n - 2)\lambda$. Thus we see that the cone over the Clifford torus $S^1(c_1) \times S^{n-2}(c_2)$, $c_1^{-1} = r_1 = \sqrt{\frac{1}{n-1}}$, $c_2^{-1} = r_2 = \sqrt{\frac{n-2}{n-1}}$, is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures satisfying at every point (3).

Example 4.2. (i) Let $\mathcal{M}$ be a surface in $\mathbb{E}^{n+1}$, $n \geq 4$, given by the immersion $f : \mathcal{M} \rightarrow \mathbb{E}^{n+1}$ and $B\mathcal{M}$ be the tangent bundle of the unit normals to $\mathcal{M}$. The hypersurface M defined by the map $\Phi_t : B\mathcal{M} \mapsto \mathbb{E}^{n+1}$, $\Phi_t(x, \xi) = F(x, t\xi) = f(x) + t\xi$, $t > 0$, is called the tube of radius t over $\mathcal{M}$. If μ_1 and μ_2 are the principal curvatures of $\mathcal{M}$ then the principal curvatures of the tube M are the following ([3]): $\lambda_1 = \frac{\mu_1}{1-t\mu_1}$, $\lambda_2 = \frac{\mu_2}{1-t\mu_2}$, $\lambda_3 = \lambda_4 = \dots = \lambda_n = -\frac{1}{t}$. Clearly, (37) holds on M ([13], Example 2).

(ii) In addition, we assume that the principal curvatures μ_1 and $\mu_2 = \mu$ of $\mathcal{M}$ are constant, and $\mu_1 = 0$ and $\mu > 0$. Moreover, let $t = \frac{n-2}{(n-1)\mu}$. Now the principal curvatures of M are the following: $\lambda_1 = 0$, $\lambda_2 = (n-1)\mu$, $\lambda_3 = -\frac{(n-1)\mu}{n-2}$ with multiplicity 1, 1 and $n-2$, respectively. Finally, if we set $\lambda = -\frac{(n-1)\mu}{n-2}$ then $\lambda_2 = -(n-2)\lambda$, and $\lambda_3 = \lambda$. Thus we see that (3) holds at every point of M .

References

- [1] A.L. Besse, Einstein Manifolds, *Ergeb. Math. Grenzgeb.*, 3. Folge, Bd. 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [2] D.E. Blair, P. Verheyen and L. Verstraelen, Hypersurfaces satisfaisant a $R \cdot C = 0$ ou $C \cdot R = 0$, *Comptes Rend. l'Acad. Bulg. Sci.* **137** (1984), 1459–1462.
- [3] T.E. Cecil and P.J. Ryan, Tight and Taut Immersions of Manifolds, Pitman, 1985.
- [4] B.-Y. Chen, Geometry of Submanifolds, Marcel Dekker, New York, 1973.
- [5] B.-Y. Chen, Riemannian submanifolds. In : Dillen F.(ed.) Handbook of Differential geometry, Vol. 1, Chap. 3, 187–418. Elsevier, Amsterdam (2000).
- [6] B.-Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, World Scientific, second edition, Series in Pure Mathematics, vol. 27, Singapore, 2014.
- [7] B.-Y. Chen and L. Verstraelen, Codimension 2 submanifolds with a quasi-umbilical normal direction, *J. Korean Math. Soc.* **13** (1976) 87–97.
- [8] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, On warped product manifolds satisfying some curvature conditions, *J. Geom. Phys.* **74** (2013), 328–341.
- [9] M. Dabrowska, F. Defever, R. Deszcz and D. Kowalczyk, Semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Euclidean spaces, *Publ. Inst. Math. (Beograd) (N.S.)* **67** (81) (2000), 103–111.
- [10] F. Defever, R. Deszcz, D. Kowalczyk and L. Verstraelen, Semisymmetry and Ricci-semisymmetry for hypersurfaces of semi-Riemannian space form, *Arab J. Math. Sc.* **6** (2000), 1–16.
- [11] F. Defever, R. Deszcz and M. Prvanović, On warped product manifolds satisfying some curvature condition of pseudosymmetry type, *Bull. Greek Math. Soc.* **36** (1994), 43–67.
- [12] J. Deprez, P. Verheyen and L. Verstraelen, Intrinsic characterizations for complex hypercylinders and complex hyperspheres, *Geom. Dedicata* **16** (1984) 217–229.
- [13] J. Deprez, R. Deszcz, L. Verstraelen and Ş. Yaprak, Hypersurfaces satisfying a pseudosymmetry condition for their Weyl conformal curvature tensor, *Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Nat.* **73** (1995), 165–181.
- [14] A. Derdziński and W. Roter, Some theorems on conformally symmetric manifolds, *Tensor (N.S.)* **32** (1978), 11–23.
- [15] A. Derdziński and W. Roter, On compact manifolds admitting indefinite metrics with parallel Weyl tensor, *J. Geom. Phys.* **58** (2008), 1137–1147.
- [16] A. Derdziński and W. Roter, Compact pseudo-Riemannian manifolds with parallel Weyl tensor, *Ann. Glob. Anal. Geom.* **37** (2010), 73–90.
- [17] R. Deszcz and M. Głogowska, Examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces, *Colloq. Math.* **94** (2002), 87–101.
- [18] R. Deszcz and M. Głogowska, Some nonsemisymmetric Ricci-semisymmetric warped product hypersurfaces, *Publ. Inst. Math. (Beograd) (N.S.)* **72** (86) (2002), 81–93.
- [19] R. Deszcz, M. Głogowska, M. Hotłoś, H. Hashiguchi and M. Yawata, On semi-Riemannian manifolds satisfying some conformally invariant curvature conditions, *Colloq. Math.* **131** (2013), 149–170.
- [20] R. Deszcz, M. Głogowska, M. Hotłoś, and K. Sawicz, A Survey on Generalized Einstein Metric Conditions, *Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin, AMS/IP Studies in Advanced Mathematics*, S.-T. Yau (series ed.), M. Plaue, A.D. Rendall and M. Scherfner (eds.), 49, 2011, 27–46.
- [21] R. Deszcz, M. Głogowska, M. Hotłoś and Z. Şentürk, On certain quasi-Einstein semisymmetric hypersurfaces, *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.* **41** (1998), 151–164.
- [22] R. Deszcz, M. Głogowska, M. Hotłoś and L. Verstraelen, On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms, *Colloq. Math.* **96** (2003), 149–166.

- [23] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz and M. Scherfner, On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type, *Kragujevac J. Math.* **35** (2011), 223–247.
- [24] R. Deszcz, A.H. Hasmani, V.G. Khambholja, and A.A. Shaikh, Curvature properties of interior black hole metric, arXiv:1401.6256v1.
- [25] R. Deszcz and M. Hotłoś, On hypersurfaces with type number two in spaces of constant curvature, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **46** (2003), 19–34.
- [26] R. Deszcz and M. Hotłoś, On some pseudosymmetry type curvature condition, *Tsukuba J. Math.* **27** (2003), 13–30.
- [27] R. Deszcz, M. Hotłoś, J. Jełowicki, H. Kundu and A.A. Shaikh, Curvature properties of Gödel metric, *Int. J. Geom. Methods Mod. Phys.* **11** (2014), 1450025 (20 pages).
- [28] R. Deszcz, M. Hotłoś and Z. Şentürk, On the equivalence of the Ricci-pseudosymmetry and pseudosymmetry, *Colloq. Math.* **79** (1999), 211–227.
- [29] R. Deszcz, M. Hotłoś and Z. Şentürk, On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces, *Soochow J. Math.* **27** (2001), 375–389.
- [30] R. Deszcz, M. Hotłoś and Z. Şentürk, Quasi-Einstein hypersurfaces in semi-Riemannian space forms, *Colloq. Math.* **89** (2001), 81–97.
- [31] R. Deszcz, M. Hotłoś and Z. Şentürk, On curvature properties of certain quasi-Einstein hypersurfaces, *Int. J. Math.* **23** (2012), 1250073, 17 pp.
- [32] R. Deszcz, M. Petrović-Torgašev, L. Verstraelen and G. Zafindratafa, On Chen ideal submanifolds satisfying some conditions of pseudo-symmetry type, *Bull. Malaysian Math. Sci. Soc.*, in print.
- [33] R. Deszcz and K. Sawicz, On some class of hypersurfaces in Euclidean spaces, *Annales Univ. Sci. Budapest. Eötvös Sect. Math.* **48** (2005), 87–98.
- [34] R. Deszcz and L. Verstraelen, Hypersurfaces of semi-Riemannian conformally flat manifolds, in: *Geometry and Topology of Submanifolds, III*, World Sci., River Edge, NJ, 1991, 131–147.
- [35] R. Deszcz, L. Verstraelen and Ş. Yaprak, Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor, *Chinese J. Math.* **22** (1994), 139–157.
- [36] R. Deszcz, L. Verstraelen and Ş. Yaprak, On 2-quasi-umbilical hypersurfaces in conformally flat spaces, *Acta Math. Hung.* **78** (1998), 45–57.
- [37] R. Deszcz and M. Yawata, On Walker type identities, in: *Pure and Applied Differential Geometry - PADGE 2007*, *Berichte aus der Mathematik*, Shaker Verlag, Aachen, 2007, 73–82.
- [38] Y. Fu, Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space, *J. Geom. Phys.* **75** (2014), 113–119.
- [39] M. Głogowska, Curvature conditions on hypersurfaces with two distinct principal curvatures, in: *Banach Center Publ., Inst. Math. Polish Acad. Sci.*, **69** (2005), 133–143.
- [40] M. Głogowska, On quasi-Einstein Cartan type hypersurfaces, *J. Geom. Phys.* **58** (2008), 599–614.
- [41] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, *Manuscripta Math.* **122** (2007), 59–72.
- [42] B. Jahanara, S. Haesen, M. Petrović-Torgašev and L. Verstraelen, On the Weyl curvature of Deszcz, *Publ. Math. Debrecen* **74** (2009), 417–431.
- [43] B. Jahanara, S. Haesen, Z. Şentürk and L. Verstraelen, On the parallel transport of the Ricci curvatures, *J. Geom. Phys.* **57** (2007), 1771–1777.
- [44] D. Kowalczyk, On the Reissner-Nordström-de Sitter type spacetimes, *Tsukuba J. Math.* **30** (2006), 363–381.
- [45] J. H. Lira and M. Soret, Examples of scalar-flat hypersurfaces in $\mathbb{R}^{n+1}$, *Manuscripta Math.* **129** (2009), 55–73.
- [46] M. Petrović, R. Rosca and L. Verstraelen, Exterior concurrent vector fields on Riemannian manifolds II, Examples of exterior concurrent vector fields on manifolds, *Soochow J. Math.* **19** (1993), 357–368.
- [47] M. Petrović-Torgašev and L. Verstraelen, On Deszcz symmetries of Wintgen ideal submanifolds, *Arch. Math. (Brno)* **44** (2008), 57–68.
- [48] K. Sawicz, Hypersurfaces in spaces of constant curvature satisfying some Ricci-type equations, *Colloq. Math.* **101** (2004), 183–201.
- [49] K. Sawicz, On some class of hypersurfaces with three principal curvatures, in: *Banach Center Publ.* **69**, *Inst. Math., Polish Acad. Sci.*, 2005, 145–156.
- [50] K. Sawicz, On curvature characterization of some hypersurfaces in spaces of constant curvature, *Publ. Inst. Math. (Beograd) (N.S.)* **79** (93) (2006), 95–107.
- [51] K. Sawicz, Curvature identities on hypersurfaces in semi-Riemannian space forms, in: *Pure and Applied Differential Geometry - PADGE 2007*, *Berichte aus der Mathematik*, Shaker Verlag, Aachen 2007, 114–122.
- [52] K. Sawicz, Curvature properties of some class of hypersurfaces in Euclidean spaces, *Publ. Inst. Math. (Beograd) (N.S.)*, in print.
- [53] L. Verstraelen, A concise mini history of geometry, *Kragujevac J. Math.* **38**(1), (2014), 5–21.
- [54] B.Y. Wu, On hypersurfaces with two distinct principal curvatures in space forms, *Proc. Indian Acad. Sci. (Math. Sci.)* **121** (2011), 435–446.