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### Research Article

### **Construction of Frames for Shift-Invariant Spaces**

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We construct a sequence  $\{\phi_i(\cdot -j) \mid j \in \mathbb{Z}, i=1,\ldots,r\}$  which constitutes a p-frame for the weighted shift-invariant space  $V_{\mu}^p(\Phi) = \{\sum_{i=1}^r \sum_{j \in \mathbb{Z}} c_i(j)\phi_i(\cdot -j) \mid \{c_i(j)\}_{j \in \mathbb{Z}} \in \ell_{\mu}^p, i=1,\ldots,r\}, p \in [1,\infty], \text{ and generates a closed shift-invariant subspace of } L_{\mu}^p(\mathbb{R}).$  The first construction is obtained by choosing functions  $\phi_i, i=1,\ldots,r$ , with compactly supported Fourier transforms  $\widehat{\phi}_i, i=1,\ldots,r$ . The second construction, with compactly supported  $\phi_i, i=1,\ldots,r$ , gives the Riesz basis.

#### 1. Introduction and Preliminaries

The shift-invariant spaces  $V_{\mu}^{p}(\Phi)$ ,  $p \in [1, \infty]$ , quoted in the abstract, are used in the wavelet analysis, approximation theory, sampling theory, and so forth. They have been extensively studied by many authors [1-18]. The aim of this paper is to construct  $V_{\mu}^{p}(\Phi)$ ,  $p \in [1, \infty]$ , spaces with specially chosen functions  $\phi_i$ , i = 1, ..., r, which generate its *p*-frame. These results extend and correct the construction obtained in [19]. For the first construction, we take functions  $\phi_i$ , i = 1, ..., r, so that the Fourier transforms are compactly supported smooth functions. Also, we derive conditions for the collection  $\{\phi_i(\cdot$  $j) \mid j \in \mathbb{Z}, i = 1, ..., r\}$  to form a Riesz basis for  $V_u^p(\Phi)$ . We note that the properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in  $V^p_\mu(\Phi)$  [20]. We generalize these results for a shift-invariant subspace of  $L^p_{\mu}(\mathbb{R}^d)$ . The second construction is obtained by choosing compactly supported functions  $\phi_i$ , i = 1, ..., r. In this way, we obtain the Riesz basis.

This paper is organized as follows. In Section 2 we quote some basic properties of certain subspaces of the weighted  $L^p$  and  $\ell^p$  spaces. In Section 3 we derive conditions for functions

of the form  $\widehat{\phi}_i(\cdot) = \theta(\cdot + k_i\pi)$ ,  $k_i \in \mathbb{Z}$ ,  $i = 1, 2, \ldots, r, r \in \mathbb{N}$ , to form a Riesz basis for  $V^p_\mu(\Phi)$ . We also show that using functions of the form  $\widehat{\phi}_i(\cdot) = \theta(\cdot + i\pi)$ ,  $i = 1, \ldots, r$ , where  $\theta$  is compactly supported smooth function whose length of support is less than or equal to  $2\pi$ , we cannot construct a p-frame for the shift-invariant space  $V^p_\mu(\Phi)$ . In Section 4 we construct a sequence  $\{\phi_i(\cdot -j) \mid j \in \mathbb{Z}, i = 0, \ldots, r\}$ , where  $r \in 2\mathbb{N}$  or  $r \in 3\mathbb{N}$ , which constitutes a p-frame for the weighted shift-invariant space  $V^p_\mu(\Phi)$ . Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust in the sense of [1]. In Section 5 we construct p-Riesz basis by using compactly supported functions  $\phi_i$ ,  $i = 1, \ldots, r$ .

#### 2. Basic Spaces

Let a function  $\omega$  be nonnegative, continuous, symmetric, and submultiplicative; that is,  $\omega(x+y) \leq \omega(x)\omega(y)$ ,  $x,y \in \mathbb{R}^d$ ; let a function  $\mu$  be  $\omega$ -moderate; that is,  $\mu(x+y) \leq C\omega(x)\mu(y)$ ,  $x,y \in \mathbb{R}^d$ . Functions  $\mu$  and  $\omega$  are called weights. We consider the weighted function spaces  $L^p_\mu$  and the weighted sequence

spaces  $\ell^p_{\mu}(\mathbb{Z}^d)$  with  $\omega$ -moderate weights  $\mu$  (see [19]). Let  $p \in [1, \infty)$ . Then (with obvious modification for  $p = \infty$ )

$$\mathcal{L}_{\omega}^{p} = \left\{ f \mid \|f\|_{\mathcal{L}_{\omega}^{p}} \right\}$$

$$= \left( \int_{[0,1]^{d}} \left( \sum_{j \in \mathbb{Z}^{d}} |f(x+j)| \omega(x+j) \right)^{p} dx \right)^{1/p}$$

$$< +\infty \right\},$$

$$W_{\omega}^{p} := \left\{ f \mid \|f\|_{W_{\omega}^{p}} \right\}$$

$$= \left( \sum_{j \in \mathbb{Z}^{d}} \sup_{x \in [0,1]^{d}} |f(x+j)|^{p} \omega(j)^{p} \right)^{1/p} < +\infty \right\}.$$

In what follows, we use the notation  $\Phi = (\phi_1, \dots, \phi_r)^T$ . Define  $\|\Phi\|_{\mathscr{H}} = \sum_{i=1}^r \|\phi_i\|_{\mathscr{H}}$ , where  $\mathscr{H} = L^p_\omega$ ,  $\mathscr{L}^p_\omega$  or  $W^p_\omega$ ,  $p \in [1,\infty]$ . With  $\mathscr{F}\phi = \widehat{\phi}$  we denote the Fourier transform of the

function  $\phi$ ; that is,  $\widehat{\phi}(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-i\pi x \cdot \xi} \, dx$ ,  $\xi \in \mathbb{R}^d$ . Let  $c = \{c_i\}_{i \in \mathbb{N}} \in \ell^p_\mu$  and  $f, g \in L^p_\omega$ ,  $p \in [1, \infty]$ . We define, as in [1], the semiconvolution f \*'c as  $(f *'c)(x) = \sum_{i \in \mathbb{Z}^d} c_i f(x-j)$ ,  $x \in \mathbb{R}^d$ , and  $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx$ .

The concept of a *p*-frame is introduced in [1].

It is said that a collection  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, ..., r\}$  is a p-frame for  $V_{\mu}^p(\Phi)$  if there exists a positive constant C (dependent upon  $\Phi$ , p, and  $\omega$ ) such that

$$C^{-1} \|f\|_{L^{p}_{\mu}}$$

$$\leq \sum_{i=1}^{r} \left\| \left\{ \int_{\mathbb{R}^{d}} f(x) \overline{\phi_{i}(x-j)} \, \mathrm{d}x \right\}_{j \in \mathbb{Z}^{d}} \right\|_{\ell^{p}_{\mu}}$$

$$\leq C \|f\|_{L^{p}_{\mu}}, \quad f \in V^{p}_{\mu}(\Phi).$$

$$(2)$$

Recall [21] that the shift-invariant spaces are defined by

$$V_{\mu}^{p}\left(\Phi\right) := \left\{ f \in L_{\mu}^{p} \mid f\left(\cdot\right) = \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}^{d}} c_{j}^{i} \phi_{i}\left(\cdot - j\right), \left\{c_{j}^{i}\right\}_{j \in \mathbb{Z}^{d}} \in \ell_{\mu}^{p}, \ i = 1, \dots, r \right\}.$$

$$(3)$$

Remark 1 (see [22]). Let  $\Phi \in W^1_\omega$  and let  $\mu$  be  $\omega$ -moderate. Then  $V^p_\mu(\Phi)$  is a subspace (not necessarily closed) of  $L^p_\mu$  and  $W^p_\mu$  for any  $p \in [1, \infty]$ . Clearly (2) implies that  $\ell^p_\mu$  and  $V^p_\mu(\Phi)$  are isomorphic Banach spaces.

Let 
$$\Phi = (\phi_1, \dots, \phi_r)^T$$
. Let 
$$\left[\widehat{\Phi}, \widehat{\Phi}\right] (\xi) = \left[ \sum_{k \in \mathbb{Z}^d} \widehat{\phi}_i \left( \xi + 2k\pi \right) \overline{\widehat{\phi}_j \left( \xi + 2k\pi \right)} \right]_{1 \le i \le r, 1 \le j \le r},$$

where we assume that  $\widehat{\phi_i}(\xi)\overline{\widehat{\phi_j}(\xi)}$  is integrable for any  $1 \leq i$ ,  $j \leq r$ . Let  $A = [a(j)]_{j \in \mathbb{Z}^d}$  be an  $r \times \infty$  matrix and  $A\overline{A^T} = [\sum_{i \in \mathbb{Z}^d} a_i(j)\overline{a_{i'}(j)}]_{1 \leq i,i' \leq r}$ . Then rank  $A = \operatorname{rank} A\overline{A^T}$ .

We recall results from [1, 19] which are needed in the sequel.

**Lemma 2** (see [1]). The following statements are equivalent.

- (1)  $\operatorname{rank}[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}^d}$  is a constant function on  $\mathbb{R}^d$ .
- (2)  $\operatorname{rank}[\widehat{\Phi}, \widehat{\Phi}](\xi)$  is a constant function on  $\mathbb{R}^d$ .
- (3) There exists a positive constant C independent of  $\xi$  such that

$$C^{-1}\left[\widehat{\Phi},\widehat{\Phi}\right](\xi) \leq \left[\widehat{\Phi},\widehat{\Phi}\right](\xi) \overline{\left[\widehat{\Phi},\widehat{\Phi}\right](\xi)^{T}}$$

$$\leq C\left[\widehat{\Phi},\widehat{\Phi}\right](\xi), \quad \xi \in \left[-\pi,\pi\right]^{d}.$$
(5)

The next theorem [19] derives necessary and sufficient conditions for an indexed family  $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, i = 1, \ldots, r\}$  to constitute a p-frame for  $V_\mu^p(\Phi)$ , which is equivalent with the closedness of this space in  $L_\mu^p$ . Thus, it is shown that under appropriate conditions on the frame vectors, there is an equivalence between the concept of p-frames, Banach frames, and the closedness of the space they generate.

**Theorem 3** (see [19]). Let  $\Phi = (\phi_1, ..., \phi_r)^T \in (W_\omega^1)^r$ ,  $p_0 \in [1, \infty]$ , and let  $\mu$  be  $\omega$ -moderate. The following statements are equivalent.

- (i)  $V_{\mu}^{p_0}(\Phi)$  is closed in  $L_{\mu}^{p_0}$ .
- (ii)  $\{\phi_i(\cdot j) \mid j \in \mathbb{Z}^d, i = 1, ..., r\}$  is a  $p_0$ -frame for  $V^{p_0}_{\mu}(\Phi)$ .
- (iii) There exists a positive constant C such that

$$C^{-1}\left[\widehat{\Phi},\widehat{\Phi}\right](\xi) \leq \left[\widehat{\Phi},\widehat{\Phi}\right](\xi) \overline{\left[\widehat{\Phi},\widehat{\Phi}\right](\xi)^{T}}$$

$$\leq C\left[\widehat{\Phi},\widehat{\Phi}\right](\xi), \quad \xi \in \left[-\pi,\pi\right]^{d}.$$
(6)

(iv) There exist positive constants  $C_1$  and  $C_2$  (depending on  $\Phi$  and  $\omega$ ) such that

$$C_{1} \| f \|_{L_{\mu}^{p_{0}}} \leq \inf_{f = \sum_{i=1}^{r} \phi_{i} *' c^{i}} \sum_{i=1}^{r} \left\| \left\{ c_{j}^{i} \right\}_{j \in \mathbb{Z}^{d}} \right\|_{\ell_{\mu}^{p_{0}}}$$

$$\leq C_{2} \| f \|_{L_{\mu}^{p_{0}}}, \quad f \in V_{\mu}^{p_{0}} \left( \Phi \right).$$

$$(7)$$

(v) There exists  $\Psi = (\psi_1, \dots, \psi_r)^T \in (W_\omega^1)^r$ , such that

$$f = \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}^{d}} \left\langle f, \psi_{i} \left( \cdot - j \right) \right\rangle \phi_{i} \left( \cdot - j \right)$$

$$=\sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^{d}}\left\langle f,\phi_{i}\left(\cdot-j\right)\right\rangle \psi_{i}\left(\cdot-j\right),\quad f\in V_{\mu}^{p_{0}}\left(\Phi\right).$$
(8)

**Corollary 4** (see [19]). Let  $\Phi = (\phi_1, \dots, \phi_r)^T \in (W_\omega^1)^r$ ,  $p_0 \in [1, \infty]$ , and let  $\mu$  be  $\omega$ -moderate.

- (i) If  $\{\phi_i(\cdot j) \mid j \in \mathbb{Z}^d, i = 1, ..., r\}$  is a  $p_0$ -frame for  $V_{\mu}^{p_0}(\Phi)$ , then the collection  $\{\phi_i(\cdot j) \mid j \in \mathbb{Z}^d, i = 1, ..., r\}$  is a p-frame for  $V_{\mu}^{p}(\Phi)$  for any  $p \in [1, \infty]$ .
- (ii) If  $V_{\mu}^{p_0}(\Phi)$  is closed in  $L_{\mu}^{p_0}$  and  $W_{\mu}^{p_0}$ , then  $V_{\mu}^{p}(\Phi)$  is closed in  $L_{\mu}^{p}$  and  $W_{\mu}^{p}$  for any  $p \in [1, \infty]$ .
- (iii) If (7) holds for  $p_0$ , then it holds for any  $p \in [1, \infty]$ .

## 3. Construction of Frames Using a Band-Limited Function

Considering the length of the support of a function  $\theta$ , we have different cases for the rank of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{i\in\mathbb{Z}}$ .

First, we consider the next claim.

Let  $\theta \in C_0^{\infty}(\mathbb{R})$  be a nonnegative function such that  $\theta(x) > 0$ ,  $x \in (-\pi, \pi)$ , and supp  $\theta \subseteq [-\pi, \pi]$ . Moreover, let

$$\widehat{\phi}_k(\xi) = \theta(\xi + k\pi), \quad k \in \mathbb{Z},$$
 (9)

and  $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$ ,  $i \in \mathbb{Z}$ ,  $r \in \mathbb{N}$ .

Then the rank of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$  is not a constant function on  $\mathbb{R}$  and it depends on  $\xi \in \mathbb{R}$ .

As a matter of fact, by the Paley-Wiener theorem,  $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_{\mu}(\mathbb{R}), i \in \mathbb{Z}$ . For any  $i \in \mathbb{Z}$ , matrix  $[\widehat{\phi}_i, \widehat{\phi}_i](\xi) = \sum_{j \in \mathbb{Z}} |\theta(\xi + i\pi + 2j\pi)|^2, \xi \in \mathbb{R}$ , has the rank 0 or 1, depending on  $\xi$ . Moreover, we have  $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](\pi) = 0$  and  $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](0) > 0$ . Because of that, the rank of the matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$  is not a constant function on  $\mathbb{R}$  and it depends on  $\xi \in \mathbb{R}$ .

**Theorem 5.** Let  $\theta \in C_0^{\infty}(\mathbb{R})$  be a non-negative function such that  $\theta(x) > 0$ ,  $x \in (-\pi - \varepsilon, \pi + \varepsilon)$ , and supp  $\theta = [-\pi - \varepsilon, \pi + \varepsilon]$ , where  $0 < \varepsilon < 1/4$ . Moreover, let

$$\widehat{\phi}_i(\xi) = \theta(\xi + k_i \pi), \quad k_i \in \mathbb{Z}, \ i = 1, 2, \dots, r, \tag{10}$$

and  $\Phi = (\phi_1, \phi_2, ..., \phi_r)^T$ .

(1) If  $|k_2 - k_1| = 2$  and  $|k_i - k_j| \ge 2$  for different  $i, j \le r$ ,

$$\operatorname{rank}\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}}=r,\quad\xi\in\mathbb{R}.\tag{11}$$

(2) If  $|k_2 - k_1| = 2$  and, at least for  $k_{i_1}$  and  $k_{i_2}$ , it holds that  $|k_{i_1} - k_{i_2}| = 1$ , where  $1 \le i_1$ ,  $i_2 \le r$ , then  $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$  is not a constant function on  $\mathbb{R}$ .

*Proof.* By the Paley-Wiener theorem,  $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_{\mu}(\mathbb{R})$ ,  $i=1,\ldots,r$ . All possible cases are described in the following lemmas.

**Lemma 6.** Let  $\Phi = (\phi_{k_1}, \phi_{k_2})^T$ ,  $k_2 - k_1 = 2$ ,  $k_1, k_2 \in \mathbb{Z}$ . The rank of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$  is a constant function on  $\mathbb{R}$  and equals 2.

Proof. We have the next two cases.

(1°) If  $\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , for matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$  we obtain  $2 \times \infty$  matrix

$$\begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \end{bmatrix}, \tag{12}$$

for some  $a^i, b^i > 0$ , i = 1, 2. It is obvious that  $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2, \xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi), \ell \in \mathbb{Z}$ .

(2°) For  $\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi]$ ,  $\ell \in \mathbb{Z}$ , there are only two nonzero values  $a^1$  and  $a^2$  which are in different columns of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{i\in\mathbb{Z}}$ . Since

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix}_{2\times\infty}, \quad (13)$$

it has the rank 2 for all  $\xi \in [-\pi + \varepsilon - k_1 \pi + 2\ell \pi, \pi - \varepsilon - k_1 \pi + 2\ell \pi], \ell \in \mathbb{Z}$ .

We conclude that the rank of matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$ ,  $\Phi=(\phi_{k_1},\phi_{k_2})^T$ ,  $k_2-k_1=2$ ,  $k_1,k_2\in\mathbb{Z}$ , is a constant function on  $\mathbb{R}$  and equals 2.

**Lemma 7.** The rank of matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  is not a constant function on  $\mathbb{R}$  if  $\Phi = (\phi_{k_1}, \phi_{k_2})^T$ ,  $k_2 - k_1 = 1$ ,  $k_1, k_2 \in \mathbb{Z}$ .

*Proof.* We have four different cases for matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$ . Suppose, without losing generality, that  $k_1 \in 2\mathbb{Z}$ .

(1°) If  $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , then

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix},$$

$$a^1, a^2, b^1 > 0,$$

$$(14)$$

and rank $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$ , for all  $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ .

(2°) For  $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$ ,  $\ell \in \mathbb{Z}$ , nonzero values  $a^1$  and  $a^2$  are in the same column of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$ . For any choice of a 2 × 2 matrix, we get that the determinant equals 0. So we obtain

$$\operatorname{rank} \begin{bmatrix} \cdots & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & \cdots \end{bmatrix} = 1, \tag{15}$$

for all  $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$ ,  $\ell \in \mathbb{Z}$ . (3°) If  $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , then

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & b^2 & a^2 & 0 & \cdots \end{bmatrix}, \quad (16)$$

for some  $a^1, a^2, b^2 > 0$ , has the rank 2, for all  $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ .

(4°) For  $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$ ,  $\ell \in \mathbb{Z}$ , there are two nonzero values  $a^1$  and  $b^2$  in different columns of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$  and the block with these elements determines the rank 2 for all  $\xi \in [-\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$ ,  $\ell \in \mathbb{Z}$ .

Considering possible cases, we conclude that  $\operatorname{rank}[\Phi(\xi+2j\pi)]_{j\in\mathbb{Z}}$ ,  $\Phi=(\phi_{k_1},\phi_{k_2})^T$ ,  $k_2-k_1=1$ ,  $k_1,k_2\in\mathbb{Z}$ , depends on  $\xi\in\mathbb{R}$  and equals 1 or 2. This rank is a nonconstant function on  $\mathbb{R}$ .

*Proof of Theorem 5.* (1) Using Lemmas 6 and 7, it is obvious that if  $|k_2-k_1|=2$  and  $|k_i-k_j|\geq 2$  for different  $i,j\leq r$ , then the position of the first non-zero element in each row of matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  is unique for each row. So the rank of matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  is a constant function on  $\mathbb{R}$  and equals r for all  $\xi\in\mathbb{R}$ .

(2) If  $|k_2-k_1|=2$  and, at least for  $k_{i_1}$  and  $k_{i_2}$ , it holds that  $|k_{i_1}-k_{i_2}|=1, 1\leq i_1, i_2\leq r$ , then, in the row with the index  $i_2$  (suppose, without losing generality, that  $i_2\in 2\mathbb{Z}+1$ ), we will have a new column with a non-zero element for  $\xi\in (-\pi-\epsilon+2\ell\pi,-\pi+\epsilon+2\ell\pi),\,\ell\in\mathbb{Z}$ , but for  $\xi\in [\epsilon+2\ell\pi,\pi-\epsilon+2\ell\pi],\,\ell\in\mathbb{Z}$ , the positions of all non-zero elements in that row will appear in the previous columns. It is obvious that the rank of the matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  depends on  $\xi\in\mathbb{R}$  and is not the same for all  $\xi\in\mathbb{R}$ .

As a consequence of Theorems 3 and 5(1), we have the following result.

**Theorem 8.** Let the functions  $\theta$  and  $\Phi$  satisfy all the conditions of Theorem 5(1). Then space  $V_{\mu}^{p}(\Phi)$  is closed in  $L_{\mu}^{p}$  for any  $p \in [1,\infty]$ , and the family  $\{\phi_{i}(\cdot -j) \mid j \in \mathbb{Z}, 1 \leq i \leq r\}$  is a p-Riesz basis for  $V_{\mu}^{p}(\Phi)$  for any  $p \in [1,\infty]$ .

The following theorem is a generalisation of Theorem 5 and can be proved in the same way, so we omit the proof.

**Theorem 9.** Let  $\theta \in C_0^{\infty}(\mathbb{R})$  be a positive function such that  $\theta(x) > 0$ ,  $x \in (a,b)$ , b > a, and supported by [a,b] where  $b-a > 2\pi$ . Moreover, let

$$\widehat{\phi_i}(\xi) = \theta(\xi + k_i \pi), \quad k_i \in \mathbb{Z}, \ i = 1, 2, \dots, r, \ r \in \mathbb{N},$$
(17)

and  $\Phi = (\phi_1, \phi_2, ..., \phi_r)^T$ .

(1) If  $|k_2 - k_1| = 2$  and  $|k_i - k_j| \ge 2$  for different  $i, j \le r$ , then

$$\operatorname{rank}\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}}=r,\quad\xi\in\mathbb{R}.\tag{18}$$

(2) If  $|k_2 - k_1| = 2$  and, at least for  $k_{i_1}$  and  $k_{i_2}$ , it holds that  $|k_{i_1} - k_{i_2}| = 1$ , where  $1 \le i_1$ ,  $i_2 \le r$ , then  $\mathrm{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$  is not a constant function on  $\mathbb{R}$ .

# **4. Construction of Frames Using Several Band-Limited Functions**

Firstly, we consider two smooth functions with proper compact supports.

**Lemma 10.** Let  $\theta \in C_0^{\infty}(\mathbb{R})$ ,  $\psi \in C_0^{\infty}(\mathbb{R})$  be positive functions such that

$$\theta(x) > 0, \quad x \in (-\varepsilon, 2\pi + \varepsilon),$$

$$\psi(x) > 0, \quad x \in (\varepsilon, 2\pi - \varepsilon),$$

$$\operatorname{supp} \theta = [-\varepsilon, 2\pi + \varepsilon],$$

$$\operatorname{supp} \psi = [\varepsilon, 2\pi - \varepsilon], \quad 0 < \varepsilon < \frac{1}{4}.$$
(19)

Moreover, let  $\widehat{\phi}_1(\xi) = \theta(\xi)$ ,  $\widehat{\phi}_2(\xi) = \psi(\xi)$ ,  $\xi \in \mathbb{R}$ , and  $\Phi = (\phi_1, \phi_2)^T$ . Then  $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 1, \xi \in \mathbb{R}$ .

*Proof.* Note that  $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_{\mu}(\mathbb{R}), i = 1, 2.$ 

We have the following two cases.

(1°) If  $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , then matrix

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a & b & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix},$$

$$a, b > 0,$$
(20)

has a constant rank equal to 1.

(2°) For  $\xi \in (\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , the rank of matrix

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & c & 0 & \cdots \\ \cdots & 0 & d & 0 & \cdots \end{bmatrix},\tag{21}$$

where c, d are non-zero values, is equal to 1. An equivalent matrix is obtained for  $\xi = \varepsilon + 2\ell\pi$  and  $\xi = -\varepsilon + 2\ell\pi$ , so we conclude that  $\mathrm{rank}[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}=1$ , for  $\xi\in[\varepsilon+2\ell\pi,2\pi-\varepsilon+2\ell\pi]$ ,  $\ell\in\mathbb{Z}$ .

Considering these two cases, the rank of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$  is a constant function on  $\mathbb{R}$ , rank  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}} = 1$ ,  $\xi \in \mathbb{R}$ .

Using functions  $\theta$  and  $\psi$  from Lemma 10, in the next lemma we construct the *p*-frame with four functions.

**Lemma 11.** Let the functions  $\theta$  and  $\psi$  satisfy all the conditions of Lemma 10. Moreover, let

$$\widehat{\phi}_{k}(\xi) = \theta(\xi + 2k\pi), \qquad \widehat{\phi}_{k+2}(\xi) = \psi(\xi + 2k\pi),$$

$$k = 0, 1,$$
(22)

and  $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$ . Then  $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$ ,  $\xi \in \mathbb{R}$ .

*Proof.* The proof is similar to the proof of Lemma 10. (1°) If  $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$ ,  $\ell \in \mathbb{Z}$ , then matrix

$$\left[\widehat{\Phi}\left(\xi + 2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, (23)$$

where  $a^i, b^i > 0$ , i = 1, 2, has a constant rank equal to 2.

(2°) For  $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi], \ell \in \mathbb{Z}$ , we have

$$\operatorname{rank}\left[\widehat{\Phi}\left(\xi + 2j\pi\right)\right]_{j \in \mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & c^{1} & 0 & \cdots \\ \cdots & 0 & 0 & d^{1} & 0 & \cdots \\ \cdots & 0 & c^{2} & 0 & 0 & \cdots \\ \cdots & 0 & d^{2} & 0 & 0 & \cdots \end{bmatrix} = 2,$$
(24)

where  $c^{i}$ ,  $d^{i} > 0$ , i = 1, 2.

We conclude that the rank of matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$  is a constant function on  $\mathbb{R}$  equal to 2.

Lemma 10 can be easily generalised for an even number of functions  $\phi_i$ , i = 0, ..., 2r - 1, with compactly supported  $\hat{\phi}_i$ , i = 0, ..., 2r - 1. The proof of the next theorem is similar to the previous proofs.

**Theorem 12.** Let the functions  $\theta$  and  $\psi$  satisfy all the conditions of Lemma 10. Moreover, let

$$\widehat{\phi}_{k}(\xi) = \theta(\xi + 2k\pi), \qquad \widehat{\phi}_{k+r}(\xi) = \psi(\xi + 2k\pi),$$

$$k = 0, \dots, r - 1, \quad r \in \mathbb{N},$$
(25)

and  $\Phi = (\phi_0, \phi_1, \dots, \phi_{2r-1})^T$ .

The following statements hold.

- (1°) rank $[\widehat{\Phi}(\xi + 2j\pi)]_{i \in \mathbb{Z}} = r \text{ for all } \xi \in \mathbb{R}.$
- (2°)  $V_{\mu}^{p}(\Phi)$  is closed in  $L_{\mu}^{p}$  for any  $p \in [1, \infty]$ .
- (3°)  $\{\phi_i(\cdot j) \mid j \in \mathbb{Z}, 0 \le i \le 2r 1\}$  is a p-frame for  $V_\mu^p(\Phi)$  for any  $p \in [1, \infty]$ .

Now we consider three functions with compact supports.

**Lemma 13.** Let the function  $\theta$  satisfies all the conditions of Lemma 10, and let  $\tau \in C_0^{\infty}(\mathbb{R})$  and  $\omega \in C_0^{\infty}(\mathbb{R})$  be positive functions such that

$$\tau(x) > 0, \quad x \in (\varepsilon, \pi - \varepsilon) \cup (\pi + \varepsilon, 2\pi - \varepsilon),$$

$$\omega(x) > 0, \quad x \in (-3\pi - \varepsilon, -\pi + \varepsilon),$$

$$\sup \tau = [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon],$$

$$\sup \omega = [-3\pi - \varepsilon, -\pi + \varepsilon], \quad 0 < \varepsilon < \frac{1}{4}.$$
(26)

Moreover, let  $\widehat{\phi}_1(\xi) = \theta(\xi)$ ,  $\widehat{\phi}_2(\xi) = \tau(\xi)$ ,  $\widehat{\phi}_3(\xi) = \omega(\xi)$ ,  $\xi \in \mathbb{R}$ , and  $\Phi = (\phi_1, \phi_2, \phi_3)^T$ . Then  $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}} = 2$ ,  $\xi \in \mathbb{R}$ .

*Proof.* We have four different forms for matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  and in all we show that the rank of matrix is equal to 2.

Now we will show all possible cases. Denote with  $a^i$ , i = 1, 2, 3, and  $b^i$ , i = 1, 2, some positive values.

(1°) Consider

$$\left[\widehat{\Phi}\left(\xi + 2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^{1} & b^{1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & a^{2} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \qquad (27)$$

$$\xi \in \left(-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi\right).$$

(2°) Consider

$$\left[\widehat{\Phi}\left(\xi+2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^2 & 0 & \cdots \\ \cdots & 0 & a^3 & 0 & 0 & \cdots \end{bmatrix},$$

$$\xi \in \left[\varepsilon+2\ell\pi, \pi-\varepsilon+2\ell\pi\right].$$
(28)

(3°) Consider

$$\left[\widehat{\Phi}\left(\xi + 2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^{1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^{2} & b^{2} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix},$$

$$\xi \in (\pi - \varepsilon + 2\ell\pi, \pi + \varepsilon + 2\ell\pi).$$
(29)

(4°) Consider

$$\left[\widehat{\Phi}\left(\xi + 2j\pi\right)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^{1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & b^{2} & 0 & \cdots \\ \cdots & 0 & a^{3} & 0 & 0 & 0 & \cdots \end{bmatrix},$$

$$\xi \in \left[\pi + \varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi\right].$$

*Remark 14.* In Lemma 13 the support of the function  $\omega$  must have an empty intersection with the supports of  $\theta$  and  $\tau$ . In the opposite case, that is, supp  $\theta \cap \text{supp } \tau \cap \text{supp } \omega \neq \emptyset$ , the rank of the matrix  $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$  is a non-constant function on  $\mathbb{R}$ .

Lemma 13 can be easily generalised for functions  $\phi_i$ , i = 0, ..., 3r - 1, with compactly supported  $\hat{\phi}_i$ , i = 0, ..., 3r - 1. The proof of the next theorem is similar to the previous proofs.

**Theorem 15.** Let the functions  $\theta$ ,  $\tau$ , and  $\omega$  satisfy all the conditions of Lemma 13. Moreover, let

$$\widehat{\phi}_{k}(\xi) = \theta(\xi + 2k\pi),$$

$$\widehat{\phi}_{k+r}(\xi) = \tau(\xi + 2k\pi),$$

$$\widehat{\phi}_{k+2r}(\xi) = \omega(\xi + 2k\pi),$$
(31)

 $k = 0, ..., r - 1, r \in \mathbb{N}$ , and  $\Phi = (\phi_0, \phi_1, ..., \phi_{3r-1})^T$ . The following statements hold.

- (1°) rank $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}} = 2r$  for all  $\xi \in \mathbb{R}$ .
- (2°)  $V_u^p(\Phi)$  is closed in  $L_u^p$  for any  $p \in [1, \infty]$ .
- (3°)  $\{\phi_i(\cdot j) \mid j \in \mathbb{Z}, \ 0 \le i \le 3r 1\}$  is a p-frame for  $V^p_\mu(\Phi)$  for any  $p \in [1, \infty]$ .

# 5. Construction of Frames of Functions with Finite Regularities and Compact Supports

We will recall the well-known construction of the *B*-spline functions in order to justify the rank properties of the corresponding matrices.

Let H(x),  $x \in \mathbb{R}$ , be the characteristic function of the semiaxis  $x \geq 0$ ; that is, H(x) = 0 if x < 0 and H(x) = 1 if  $x \geq 0$  (Heaviside's function). We construct a sequence  $\{\phi_n\}_{n \in \mathbb{N}}$  in the following way. Let  $\phi_1(x) := (H(x) - H(x-a))/a$ , a > 0,  $\phi_2 := \phi_1 * \phi_1$ ,  $\phi_3 := \phi_1 * \phi_1 * \phi_1$ , ...; that is,

$$\phi_n := \underbrace{\phi_1 * \phi_1 * \cdots * \phi_1}_{n-1 \text{ times}}, \quad n \in \mathbb{N},$$
(32)

where \* denotes the convolution of the functions. We obtain

$$\phi_{2}(x) = \frac{1}{a^{2}} (xH(x) - 2(x - a)H(x - a) + (x - 2a)H(x - 2a)),$$

$$+ (x - 2a)H(x - 2a)),$$

$$\phi_{3}(x) = \frac{1}{2!a^{3}} (x^{2}H(x) - 3(x - a)^{2}H(x - a) + 3(x - 2a)^{2} + (x - 2a) - (x - 3a)^{2}H(x - 3a)),$$

$$\psi_{4}(x) = \frac{1}{3!a^{4}} (x^{3}H(x) - 4(x - a)^{3}H(x - a) + 6(x - 2a)^{3}H(x - 2a) - 4(x - 3a)^{3} + (x - 3a) + (x - 4a)^{3}H(x - 4a)).$$
(33)

Continuing in this manner, for all  $n \in \mathbb{N}$ , we have

$$\phi_{n}(x) = \frac{1}{a^{n}(n-1)!} \times \left(\binom{n}{0}x^{n-1}H(x) - \binom{n}{1}(x-a)^{n-1} \times H(x-a) + \binom{n}{2}(x-2a)^{n-1} \times H(x-2a) - \binom{n}{3}(x-3a)^{n-1}H(x-3a) + \dots + (-1)^{n-1}\binom{n}{n-1}(x-(n-1)a)^{n-1} \times H(x-(n-1)a) + (-1)^{n}\binom{n}{n} \times H(x-(n-1)a) + (-1)^{n}\binom{n}{n}$$

$$\times (x-na)^{n-1}H(x-na)\right). \tag{34}$$

Calculating the Fourier transform of functions  $\phi_n$ ,  $n \in \mathbb{N}$ , we get

$$\widehat{\phi}_{1}(\xi) = \frac{-i}{a} \nu \cdot p \cdot \left(\frac{1}{\xi}\right) \left(e^{ia\xi} - 1\right),$$

$$\widehat{\phi}_{2}(\xi) = \frac{(-i)^{2}}{a^{2}} \nu \cdot p \cdot \left(\frac{1}{\xi^{2}}\right) \left(e^{ia\xi} - 1\right)^{2},$$

$$\widehat{\phi}_{3}(\xi) = \frac{(-i)^{3}}{a^{3}} \nu \cdot p \cdot \left(\frac{1}{\xi^{3}}\right) \left(e^{ia\xi} - 1\right)^{3}.$$
(35)

Continuing in this manner, we obtain  $\widehat{\phi}_n(\xi) = (-i)^n/a^n v \cdot p \cdot (1/\xi^n)(e^{ia\xi} - 1)^n$ ,  $n \in \mathbb{N}$ , where  $v \cdot p \cdot$  denotes the principal value.

Let  $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$ ,  $r \in \mathbb{N}$ . matrix  $[\widehat{\Phi}(\xi + 2j\pi)]_{j \in \mathbb{Z}}$  has for all  $\xi \in \mathbb{R}$  the same rank as matrix

 $R(\xi)$ 

$$=\begin{bmatrix} \cdots & \alpha_{-4\pi}\beta_{-4\pi} & \alpha_{-2\pi}\beta_{-2\pi} & \alpha_0\beta_0 & \alpha_{2\pi}\beta_{2\pi} & \alpha_{4\pi}\beta_{4\pi} & \cdots \\ \cdots & \alpha_{-4\pi}^2\beta_{-4\pi}^2 & \alpha_{-2\pi}^2\beta_{-2\pi}^2 & \alpha_0^2\beta_0^2 & \alpha_{2\pi}^2\beta_{2\pi}^2 & \alpha_{4\pi}^2\beta_{4\pi}^2 & \cdots \\ \cdots & \alpha_{-4\pi}^3\beta_{-4\pi}^3 & \alpha_{-2\pi}^3\beta_{-2\pi}^3 & \alpha_0^3\beta_0^3 & \alpha_{2\pi}^3\beta_{2\pi}^3 & \alpha_{4\pi}^3\beta_{4\pi}^3 & \cdots \\ \cdots & \alpha_{-4\pi}^4\beta_{-4\pi}^4 & \alpha_{-2\pi}^4\beta_{-2\pi}^4 & \alpha_0^4\beta_0^4 & \alpha_{2\pi}^4\beta_{2\pi}^4 & \alpha_{4\pi}^4\beta_{4\pi}^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \alpha_{-4\pi}^r\beta_{-4\pi}^r & \alpha_{-2\pi}^r\beta_{-2\pi}^r & \alpha_0^r\beta_0^r & \alpha_{2\pi}^r\beta_{2\pi}^r & \alpha_{4\pi}^r\beta_{4\pi}^r & \cdots \end{bmatrix},$$
(36)

where  $\alpha_k^m = v \cdot p \cdot (1/(\xi - k))^m$  and  $\beta_k^m = (e^{ia(\xi - k)} - 1)^m$ . Since rank  $R(\xi) = r, \xi \in \mathbb{R}$ , we have the next result.

**Theorem 16.** Let  $\Phi = (\phi_k, \phi_{k+1}, \dots, \phi_{k+(r-1)})^T$ , for  $k \in \mathbb{Z}$ ,  $r \in \mathbb{N}$ . Then  $V_{\mu}^p(\Phi)$  is closed in  $L_{\mu}^p$  for any  $p \in [1, \infty]$  and  $\{\phi_{k+s}(\cdot - j) \mid j \in \mathbb{Z}, \ 0 \le s \le r - 1\}$  is a p-Riesz basis for  $V_{\mu}^p(\Phi)$  for any  $p \in [1, \infty]$ .

Remark 17. Let k be a positive integer. We refer to [23] for the  $L^p$ -approximation order k. Shift-invariant spaces generated by a finite number of compactly supported functions in  $L^p(\mathbb{R})$ ,  $1 \le p \le \infty$ , were studied in [23] by Jia, who gave a characterization of the approximation order providing such shift-invariant spaces. Theorem 3 in [23] shows that the shift-invariant space generated with the family of splines, which we constructed in Section 5, provides  $L^p$ -approximation order k.

Remark 18. (1) We refer to [3, 20] for the  $\gamma$ -dense set  $X = \{x_j \mid j \in J\}$ . Let  $\phi_k(x) = \mathcal{F}^{-1}(\theta(\cdot - k\pi))(x), x \in \mathbb{R}$ . Following the notation of [20], we put  $\psi_{x_j} = \phi_{x_j}$ , where  $\{x_j \mid j \in J\}$  is  $\gamma$ -dense set determined by  $f \in V^2(\phi) = V^2(\mathcal{F}^{-1}(\theta))$ . Theorems 3.1, 3.2, and 4.1 in [20] give conditions and explicit form of  $C_p > 0$  and  $c_p > 0$  such that inequality  $c_p \|f\|_{L^p_\mu} \leq (\sum_{j \in J} |\langle f, \psi_{x_j} \rangle \mu(x_j)|^p)^{1/p} \leq C_p \|f\|_{L^p_\mu}$  holds. This inequality guarantees the feasibility of a stable and continuous reconstruction algorithm in the signal spaces  $V^p_\mu(\Phi)$  [20].

- (2) Since the spectrum of the Gram matrix  $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ , where  $\Phi$  is defined in Theorem 16, is bounded and bounded away from zero (see [7]), it follows that the family  $\{\Phi(\cdot j) \mid j \in \mathbb{Z}\}$  forms a p-Riesz basis for  $V_u^p(\Phi)$ .
- (3) Frames of the above sections may be useful in applications since they satisfy assumptions of Theorems 3.1 and 3.2 in [4]. They show that error analysis for sampling and reconstruction can be tolerated or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions  $\phi_{k_1}, \ldots, \phi_{k_r}$ .

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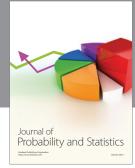
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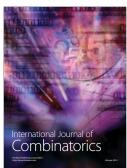










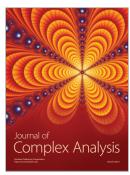




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