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Some common fixed point theorems for a family of non-self mappings in cone metric spaces

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Abstract

Some common fixed point theorems for a family of non-self mappings defined on a closed subset of a metrically convex cone metric space (over the cone which is not necessarily normal) are obtained which generalize earlier results due to Imdad *et al.* and Janković *et al.*

MSC: 47H10; 54H25

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1 Introduction and preliminaries

The existing literature of fixed point theory contains many results enunciating fixed point theorems for self-mappings in metric and Banach spaces. Recently, Huang and Zhang [1] have replaced the real numbers by ordering Banach space and defining cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians; see [2–17]. However, fixed point theorems for non-self mappings are not frequently discussed and so they form a natural subject for further investigation. The study of fixed point theorems for non-self mappings in metrically convex metric spaces was initiated by Assad and Kirk [18]. Recently, Janković *et al.* [10] obtained a fixed point theorem for two non-self mappings in cone metric spaces. Motivated by Janković *et al.* [10], we prove some common fixed point theorems for a family of non-self mappings on cone metric spaces in which the cone need not be normal.

Consistent with Huang and Zhang [1], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (interior of P).

Definition 1.1 [1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 1.2 [1] Let (X,d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n,m > N$, $d(x_n,x_m) \ll c$;

(f) a convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n,x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if for every Cauchy sequence in X, it is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n,x) \to \theta$ as $n \to \infty$. It is a Cauchy sequence if and only if $d(x_n,x_m) \to \theta$ $(n,m \to \infty)$.

Remark 1.1 [19] Let E be an ordered Banach (normed) space. Then c is an interior point of P, if and only if $[-c,c]$ is a neighborhood of θ.

Corollary 1.1 [9] (1) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Indeed, $c - a = (c - b) + (b - a) \succeq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(2) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

Indeed, $c - a = (c - b) + (b - a) \succeq c - b$ implies $[-(c - a), c - a] \supseteq [-(c - b), c - b]$.

(3) *If $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.*

Remark 1.2 [11] If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists an n_0 such that for all $n > n_0$, we have $a_n \ll c$.

Remark 1.3 [11] If E is a real Banach space with cone P and if $a \preceq ka$, where $a \in P$ and $0 < k < 1$, then $a = \theta$.

We find it convenient to introduce the following definition.

Definition 1.3 [11] Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X, and let $f,g : C \to X$ be non-self mappings. Denote, for $x,y \in C$,

$$M_1^{f,g} = \left\{ d(gx,gy), d(fx,gx), d(fy,gy), \frac{d(fx,gy) + d(fy,gx)}{2} \right\}. \tag{1.1}$$

Then f is called a generalized g_{M_1}-contractive mapping of C into X if, for some $\lambda \in (0, \sqrt{2} - 1)$, there exists $u(x,y) \in M_1^{f,g}$ such that for all $x,y \in C$,

$$d(fx,fy) \preceq \lambda u(x,y). \tag{1.2}$$

Definition 1.4 [2] Let f and g be self-maps of a set X (*i.e.*, $f,g: X \to X$). If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. Self-maps f and g are said to be coincidentally commuting if they commute at their coincidence point; *i.e.*, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

2 Main results

Recently, Janković *et al.* [10] proved some fixed point theorems for a pair of non-self mappings defined on a nonempty closed subset of complete metrically convex cone metric spaces with new contractive conditions.

Theorem 2.1 [10] *Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $f,g: C \to X$ are such that f is a generalized g_{M_1}-contractive mapping of C into X, and

(i) $\partial C \subseteq gC, fC \cap C \subseteq gC$,

(ii) $gx \in \partial C \Rightarrow fx \in C$,

(iii) *gC is closed in X.*

Then the pair (f,g) has a coincidence point. Moreover, if (f,g) are coincidentally commuting, then f and g have a unique common fixed point.

The purpose of this paper is to extend the above theorem for a family of non-self mappings in cone metric spaces. We begin with the following definition.

Definition 2.1 Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X, and let $\{F_n\}_{n=1}^{\infty}, S, T: C \to X$ be non-self mappings. Denote, for $x,y \in C$,

$$M_1^{F_n,S,T} = \left\{ d(Tx,Sy), d(Tx,F_ix), d(Sy,F_jy), \frac{d(Tx,F_jy) + d(F_ix,Sy)}{2} \right\}, \tag{2.1}$$

where $i = 2n-1, j = 2n$ for some $n \in N$. Then (F_i,F_j) is called a pair of generalized $(T,S)_{M_1}$-contractive mappings of C into X if for some $\lambda \in (0,1)$ there exists $u(x,y) \in M_1^{F_n,S,T}$ such that for all $x,y \in C$ with $x \neq y$,

$$d(F_ix,F_jy) \preceq \lambda u(x,y). \tag{2.2}$$

Notice that by setting $F_i = F_j = f$, $T = S = g$ and $\lambda \in (0, \sqrt{2}-1)$ in (2.1), one deduces a slightly generalized form of (1.1).

We state and prove our main result as follows.

Theorem 2.2 *Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $F_n, S, T : C \to X$ are such that (F_i, F_j) is a pair of generalized $(T,S)_{M_1}$-contractive mappings of C into X for all $i = 2n-1$, $j = 2n$ ($n \in N$), and

(I) $\partial C \subseteq SC \cap TC$, $F_iC \cap C \subseteq SC$, $F_jC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $F_ix \in C$, $Sx \in \partial C$ *implies that* $F_jx \in C$,

(III) SC *and* TC (*or* F_iC *and* F_jC) *are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence,*

(V) (F_j, S) *has a point of coincidence.*

Moreover, if (F_i, T) and (F_j, S) are coincidentally commuting pairs, then $\{F_n\}_{n=1}^{\infty}$, S and T have a unique common fixed point.

Proof Let $x \in \partial C$ be arbitrary. Then (due to $\partial C \subseteq TC$) there exists a point $x_0 \in C$ such that $x = Tx_0$. Since $Tx_0 \in \partial C$, from (I) and (II), we have $F_1x_0 \in F_1C \cap C \subseteq SC$. Thus, there exists $x_1 \in C$ such that $y_1 = Sx_1 = F_1x_0 \in C$. Since $y_1 = F_1x_0$, there exists a point $y_2 = F_2x_1$ such that

$$d(y_1, y_2) = d(F_1x_0, F_2x_1).$$

Suppose $y_2 \in C$. Then $y_2 \in F_2C \cap C \subseteq TC$, which implies that there exists a point $x_2 \in C$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin C$, then there exists a point $p \in \partial C$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \partial C \subseteq TC$, there exists a point $x_2 \in C$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 = F_3x_2$ be such that $d(y_2, y_3) = d(F_2x_1, F_3x_2)$. Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

(a) $y_{2n} = F_{2n}x_{2n-1}$, $y_{2n+1} = F_{2n+1}x_{2n}$,

(b) $y_{2n} \in C$ implies that $y_{2n} = Tx_{2n}$ or $y_{2n} \notin C$ implies that $Tx_{2n} \in \partial C$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}),$$

(c) $y_{2n+1} \in C$ implies that $y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin C$ implies that $Sx_{2n+1} \in \partial C$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$P_0 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\},$$
$$P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\},$$
$$Q_0 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\},$$
$$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.$$

Note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$, as if $Tx_{2n} \in P_1$, then $y_{2n} \neq Tx_{2n}$ and one infers that $Tx_{2n} \in \partial C$, which implies that $y_{2n+1} = F_{2n+1}x_{2n} \in C$. Hence $y_{2n+1} = Sx_{2n+1} \in Q_0$. Similarly, one can argue that $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0$, then from (2.2)

$$d(Tx_{2n}, Sx_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \preceq \lambda u_{2n-1},$$

where

$$\begin{aligned} u_{2n-1} \in & \Big\{ d(Sx_{2n-1}, Tx_{2n}), d(Sx_{2n-1}, F_{2n}x_{2n-1}), d(Tx_{2n}, F_{2n+1}x_{2n}), \\ & \frac{d(Tx_{2n}, F_{2n}x_{2n-1}) + d(Sx_{2n-1}, F_{2n+1}x_{2n})}{2} \Big\} \\ = & \Big\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2} \Big\}. \end{aligned}$$

Clearly, there are infinitely many n such that at least one of the following three cases holds:

(1) $d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(y_{2n-1}, y_{2n}) = \lambda d(Sx_{2n-1}, Tx_{2n})$;
(2) $d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(y_{2n}, y_{2n+1})$ implies that $d(Tx_{2n}, Sx_{2n+1}) = \theta \preceq \lambda d(Sx_{2n-1}, Tx_{2n})$;
(3) $d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda \frac{d(y_{2n-1}, y_{2n+1})}{2} \preceq \frac{\lambda}{2} d(y_{2n-1}, y_{2n}) + \frac{1}{2} d(y_{2n}, y_{2n+1})$ implies that $d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n})$.

From (1), (2), (3) it follows that

$$d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n}). \tag{2.3}$$

Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n+1}, Tx_{2n+2}) = d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \preceq \lambda d(Tx_{2n}, Sx_{2n+1}). \tag{2.4}$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0$, we have

$$d(Sx_{2n-1}, Tx_{2n}) = d(F_{2n-1}x_{2n-2}, F_{2n}x_{2n-1}) \preceq \lambda d(Tx_{2n-2}, Sx_{2n-1}). \tag{2.5}$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1$, then $Sx_{2n+1} \in Q_1$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}), \tag{2.6}$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \preceq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) \tag{2.7}$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \preceq d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}). \tag{2.8}$$

Now, proceeding as in Case 1, we have that (2.3) holds.

If $(Sx_{2n+1}, Tx_{2n+2}) \in Q_1 \times P_0$, then $Tx_{2n} \in P_0$. We show that

$$d(Sx_{2n+1}, Tx_{2n+2}) \preceq \lambda d(Tx_{2n}, Sx_{2n-1}). \tag{2.9}$$

Using (2.6), we get

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\preceq d(Sx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}) \\ &= d(Tx_{2n}, y_{2n+1}) - d(Tx_{2n}, Sx_{2n+1}) + d(y_{2n+1}, Tx_{2n+2}). \end{aligned} \tag{2.10}$$

By noting that $Tx_{2n+2}, Tx_{2n} \in P_0$, one can conclude that

$$d(y_{2n+1}, Tx_{2n+2}) = d(y_{2n+1}, y_{2n+2}) = d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \preceq \lambda d(Tx_{2n}, Sx_{2n+1}) \tag{2.11}$$

and

$$d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n}), \tag{2.12}$$

in view of Case 1.

Thus,

$$d(Sx_{2n+1}, Tx_{2n+2}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n}) - (1-\lambda) d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n}),$$

and we proved (2.9).

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0$, then $Sx_{2n-1} \in Q_0$. We show that

$$d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n-2}). \tag{2.13}$$

Since $Tx_{2n} \in P_1$, then

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}). \tag{2.14}$$

From this, we get

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\preceq d(Tx_{2n}, y_{2n}) + d(y_{2n}, Sx_{2n+1}) \\ &= d(Sx_{2n-1}, y_{2n}) - d(Sx_{2n-1}, Tx_{2n}) + d(y_{2n}, Sx_{2n+1}). \end{aligned} \tag{2.15}$$

By noting that $Sx_{2n+1}, Sx_{2n-1} \in Q_0$, one can conclude that

$$d(y_{2n}, Sx_{2n+1}) = d(y_{2n}, y_{2n+1}) = d(F_{2n+1}x_{2n}, F_{2n}x_{2n-1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n}) \tag{2.16}$$

and

$$d(Sx_{2n-1}, y_{2n}) = d(y_{2n-1}, y_{2n}) = d(F_{2n-1}x_{2n-2}, F_{2n}x_{2n-1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n-2}), \tag{2.17}$$

in view of Case 1.

Thus,

$$d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n-2}) - (1-\lambda) d(Sx_{2n-1}, Tx_{2n}) \preceq \lambda d(Sx_{2n-1}, Tx_{2n-2}),$$

and we proved (2.13).

Similarly, if $(Sx_{2n+1}, Tx_{2n+2}) \in Q_0 \times P_1$, then $Tx_{2n+2} \in P_1$, and

$$d(Sx_{2n+1}, Tx_{2n+2}) + d(Tx_{2n+2}, y_{2n+2}) = d(Sx_{2n+1}, y_{2n+2}).$$

From this, we have

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\preceq d(Sx_{2n+1}, y_{2n+2}) + d(y_{2n+2}, Tx_{2n+2}) \\ &\preceq d(Sx_{2n+1}, y_{2n+2}) + d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}) \\ &= 2d(Sx_{2n+1}, y_{2n+2}) - d(Sx_{2n+1}, Tx_{2n+2}). \end{aligned}$$

This implies that $d(Sx_{2n+1}, Tx_{2n+2}) \preceq d(Sx_{2n+1}, y_{2n+2})$.

By noting that $Sx_{2n+1} \in Q_0$, one can conclude that

$$\begin{aligned} d(Sx_{2n+1}, Tx_{2n+2}) &\preceq d(Sx_{2n+1}, y_{2n+2}) \\ &= d(F_{2n+1}x_{2n}, F_{2n+2}x_{2n+1}) \preceq \lambda d(Tx_{2n}, Sx_{2n+1}), \end{aligned} \tag{2.18}$$

in view of Case 1.

Thus, in all Cases 1-3, there exists $w_{2n} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda w_{2n}$$

and there exists $w_{2n+1} \in \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}$ such that

$$d(Sx_{2n+1}, Tx_{2n+2}) \preceq \lambda w_{2n+1}.$$

Following the procedure of Assad and Kirk [18], it can easily be shown by induction that, for $n \geq 1$, there exists $w_2 \in \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}$ such that

$$d(Tx_{2n}, Sx_{2n+1}) \preceq \lambda^{n-\frac{1}{2}} w_2 \quad \text{and} \quad d(Sx_{2n+1}, Tx_{2n+2}) \preceq \lambda^n w_2. \tag{2.19}$$

From (2.19) and by the triangle inequality, for $n > m$, we have

$$\begin{aligned} d(Tx_{2n}, Sx_{2m+1}) &\preceq d(Tx_{2n}, Sx_{2n-1}) + d(Sx_{2n-1}, Tx_{2n-2}) + \cdots + d(Tx_{2m+2}, Sx_{2m+1}) \\ &\preceq \left(\lambda^m + \lambda^{m+\frac{1}{2}} + \cdots + \lambda^{n-1}\right) w_2 \preceq \frac{\lambda^m}{1-\sqrt{\lambda}} w_2 \to \theta, \quad \text{as } m \to \infty. \end{aligned}$$

From Remark 1.2 and Corollary 1.1(1), $d(Tx_{2n}, Sx_{2m+1}) \ll c$.

Thus, the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \ldots\}$ is a Cauchy sequence. Then, as noted in [20], there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_0 or Q_0, respectively, having as a limit point z. Furthermore, subsequences

$\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in C$ as C is a closed subset of a complete cone metric space (X,d). We assume that there exists a subsequence $\{Tx_{2n_k}\} \subseteq P_0$ for each $k \in N$, and TC as well as SC are closed in X. Since $\{Tx_{2n_k}\}$ is a Cauchy sequence in TC, it converges to a point $z \in TC$. Let $w \in T^{-1}z$, then $Tw = z$. Similarly, $\{Sx_{2n_k+1}\}$ being a subsequence of the Cauchy sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \ldots, Sx_{2n-1}, Tx_{2n}, Sx_{2n-1}, \ldots\}$ also converges to z as SC is closed. Using (2.2), one can write

$$d(F_i w, z) \preceq d(F_i w, F_j x_{2n_k-1}) + d(F_j x_{2n_k-1}, z) \preceq \lambda u_{2n_k-1} + d(F_j x_{2n_k-1}, z),$$

where

$$\begin{aligned} u_{2n_k-1} \in {} & \biggl\{ d(Tw, Sx_{2n_k-1}), d(Tw, F_i w), d(Sx_{2n_k-1}, F_j x_{2n_k-1}), \\ & \frac{d(Tw, F_j x_{2n_k-1}) + d(F_i w, Sx_{2n_k-1})}{2} \biggr\} \\ = {} & \biggl\{ d(z, Sx_{2n_k-1}), d(z, F_i w), d(Sx_{2n_k-1}, F_j x_{2n_k-1}), \frac{d(z, F_j x_{2n_k-1}) + d(F_i w, Sx_{2n_k-1})}{2} \biggr\} \end{aligned}$$

for any odd integer $i \in N$ and even integer $j \in N$.

Let $\theta \ll c$. Clearly at least one of the following four cases holds for infinitely many n.

(1) $d(F_i w, z) \preceq \lambda d(z, Sx_{2n_k-1}) + d(F_j x_{2n_k-1}, z) \ll \lambda \frac{c}{2\lambda} + \frac{c}{2} = c$;

(2) $d(F_i w, z) \preceq \lambda d(z, F_i w) + d(F_j x_{2n_k-1}, z) \Rightarrow d(F_i w, z) \preceq \frac{1}{1-\lambda} d(F_j x_{2n_k-1}, z) \ll \frac{1}{1-\lambda}(1-\lambda)c = c$;

(3)

$$\begin{aligned} d(F_i w, z) &\preceq \lambda d(Sx_{2n_k-1}, F_j x_{2n_k-1}) + d(F_j x_{2n_k-1}, z) \\ &\preceq \lambda \bigl(d(Sx_{2n_k-1}, z) + d(z, F_j x_{2n_k-1}) \bigr) + d(F_j x_{2n_k-1}, z) \\ &\preceq (\lambda+1) d(F_j x_{2n_k-1}, z) + \lambda d(Sx_{2n_k-1}, z) \ll (\lambda+1) \frac{c}{2(\lambda+1)} + \lambda \frac{c}{2\lambda} = c; \end{aligned}$$

(4)

$$\begin{aligned} d(F_i w, z) &\preceq \lambda \frac{d(z, F_j x_{2n_k-1}) + d(F_i w, Sx_{2n_k-1})}{2} + d(F_j x_{2n_k-1}, z) \\ &\preceq \lambda \frac{d(z, F_j x_{2n_k-1}) + d(z, Sx_{2n_k-1})}{2} + \frac{1}{2} d(F_i w, z) + d(F_j x_{2n_k-1}, z) \\ \Rightarrow \quad d(F_i w, z) &\preceq (2+\lambda) d(F_j x_{2n_k-1}, z) + \lambda d(z, Sx_{2n_k-1}) \\ &\ll (2+\lambda) \frac{c}{2(2+\lambda)} + \lambda \frac{c}{2\lambda} = c. \end{aligned}$$

In all cases, we obtain $d(F_i w, z) \ll c$ for each $c \in \operatorname{int} P$. Using Corollary 1.1(3), it follows that $d(F_i w, z) = \theta$ or $F_i w = z$. Thus, $F_i w = z = Tw$, that is, z is a coincidence point of F_i, T for any odd integer $i \in N$.

Further, since the Cauchy sequence $\{T_{x_{2n_k}}\}$ converges to $z \in C$ and $z = F_i w$, $z \in F_i C \cap C \subseteq SC$, there exists $v \in C$ such that $Sv = z$. Again, using (2.2), we get

$$d(Sv, F_j v) = d(z, F_j v) = d(F_i w, F_j v) \preceq \lambda u,$$

where

$$u \in \left\{ d(Tw, Sv), d(Tw, F_i w), d(Sv, F_j v), \frac{d(Tw, F_j v) + d(F_i w, Sv)}{2} \right\}$$
$$= \left\{ \theta, \theta, d(Sv, F_j v), \frac{d(z, F_j v) + \theta}{2} \right\} = \left\{ \theta, d(Sv, F_j v), \frac{d(Sv, F_j v)}{2} \right\}$$

for any odd integer $i \in N$ and even integer $j \in N$.

Hence, we get the following cases:

$$d(Sv, F_j v) \preceq \lambda\theta = \theta, \qquad d(Sv, F_j v) \preceq \lambda d(Sv, F_j v) \quad \text{and}$$
$$d(Sv, F_j v) \preceq \frac{\lambda}{2} d(Sv, F_j v) \preceq \lambda d(Sv, F_j v).$$

Using Remark 1.3 and Corollary 1.1(3), it follows that $Sv = F_j v$; therefore, $Sv = z = F_j v$, that is, z is a coincidence point of (F_j, S) for any even integer $j \in N$.

In case $F_i C$ and $F_j C$ are closed in X, then $z \in F_i C \cap C \subseteq SC$ or $z \in F_j C \cap C \subseteq TC$. The analogous arguments establish (IV) and (V). If we assume that there exists a subsequence $\{Sx_{2n_k+1}\} \subseteq Q_0$ with TC as well as SC closed in X, then noting that $\{Sx_{2n_k+1}\}$ is a Cauchy sequence in SC, foregoing arguments establish (IV) and (V).

Suppose now that (F_i, T) and (F_j, S) are coincidentally commuting pairs, then

$$z = F_i w = Tw \quad \Rightarrow \quad F_i z = F_i Tw = TF_i w = Tz \quad \text{and}$$
$$z = F_j v = Sv \quad \Rightarrow \quad F_j z = F_j Sv = SF_j v = Sz.$$

Then, from (2.2),

$$d(F_i z, z) = d(F_i z, F_j v) \preceq \lambda u,$$

where

$$u \in \left\{ d(Sv, Tz), d(Tz, F_i z), d(Sv, F_j v), \frac{d(Tz, F_j v) + d(Sv, F_i z)}{2} \right\}$$
$$= \left\{ d(z, F_i z), d(z, z), \frac{d(F_i z, z) + d(z, F_i z)}{2} \right\} = \left\{ d(z, F_i z), \theta \right\}.$$

Hence, we get the following cases:

$$d(F_i z, z) \preceq \lambda d(z, F_i z) \quad \Rightarrow \quad d(F_i z, z) = 0,$$
$$d(F_i z, z) \preceq \lambda\theta = \theta \quad \Rightarrow \quad d(F_i z, z) = 0.$$

Using Remark 1.3 and Corollary 1.1(3), it follows that $F_i z = z$. Thus, $F_i z = z = Tz$.

Similarly, we can prove $F_j z = z = Sz$. Therefore $z = F_i z = F_j z = Sz = Tz$, that is, z is a common fixed point of F_n, S and T.

The uniqueness of the common fixed point follows easily from (2.2). □

Example 2.1 Let $E = C^1([0,1],R)$, $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0,1]\}$, $X = [0,+\infty)$, $C = [0,2]$ and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$, where $\varphi \in P$ is a fixed function, *e.g.*, $\varphi(t) = e^t$. Then (X,d) is a complete cone metric space with a non-normal cone having the nonempty interior. Define F_i, F_j, S and $T : C \to X$ as

$$F_i x = x + \frac{4}{5}, \quad i = 2n-1, \qquad F_j x = x^2 + \frac{4}{5}, \quad j = 2n,$$
$$Tx = 5x \quad \text{and} \quad Sx = 5x^2, \quad x \in C.$$

Since $\partial C = \{0,2\}$. Clearly, for each $x \in C$ and $y \notin C$, there exists a point $z = 2 \in \partial C$ such that $d(x,z) + d(z,y) = d(x,y)$. Further, $SC \cap TC = [0,20] \cap [0,10] = [0,10] \supset \{0,2\} = \partial C$, $F_iC \cap C = [\frac{4}{5}, \frac{14}{5}] \cap [0,2] = [\frac{4}{5}, 2] \subset SC$, $F_jC \cap C = [\frac{4}{5}, \frac{24}{5}] \cap [0,2] = [\frac{4}{5}, 2] \subset TC$, and SC, TC, F_iC and F_jC are closed in X.

Also,

$$T0 = 0 \in \partial C \quad \Rightarrow \quad F_i 0 = \frac{4}{5} \in C, \qquad S0 = 0 \in \partial C \quad \Rightarrow \quad F_j 0 = \frac{4}{5} \in C,$$
$$T\left(\frac{2}{5}\right) = 2 \in \partial C \quad \Rightarrow \quad F_i\left(\frac{2}{5}\right) = \frac{6}{5} \in C,$$
$$S\left(\sqrt{\frac{2}{5}}\right) = 2 \in \partial C \quad \Rightarrow \quad F_j\left(\sqrt{\frac{2}{5}}\right) = \frac{6}{5} \in C.$$

Moreover, for each $x, y \in C$,

$$d(F_i x, F_j y) = |x - y^2|\varphi = \frac{1}{5} d(Tx, Sy)$$

that is, (2.2) is satisfied with $\lambda = \frac{1}{5}$.

Evidentially, $1 = T(\frac{1}{5}) = F_i(\frac{1}{5}) \neq \frac{1}{5}$ and $1 = S(\frac{1}{\sqrt{5}}) = F_j(\frac{1}{\sqrt{5}}) \neq \frac{1}{\sqrt{5}}$. Notice that two separate coincidence points are not common fixed points as $F_iT(\frac{1}{5}) \neq TF_i(\frac{1}{5})$ and $SF_j(\frac{1}{\sqrt{5}}) \neq F_jS(\frac{1}{\sqrt{5}})$, which shows the necessity of coincidentally commuting property in Theorem 2.2.

Next, we furnish an illustrative example in support of our result. In doing so, we are essentially inspired by Imdad and Kumar [21].

Example 2.2 Let $E = C^1([0,1],R)$, $P = \{\varphi \in E : \varphi(t) \geq 0, t \in [0,1]\}$, $X = [1,+\infty)$, $C = [1,3]$ and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$, where $\varphi \in P$ is a fixed function, *e.g.*, $\varphi(t) = e^t$. Then (X,d) is a complete cone metric space with a non-normal cone having the nonempty interior. Define F_i, F_j, S and $T : C \to X$ as

$$F_i x = \begin{cases} \frac{x^2-1+n}{n} & \text{if } 1 \leq x \leq 2, \\ \frac{n+1}{n} & \text{if } 2 < x \leq 3, \end{cases} \quad i = 2n-1\ (n \geq 1), \qquad Tx = \begin{cases} 4x^4 - 3 & \text{if } 1 \leq x \leq 2, \\ 13 & \text{if } 2 < x \leq 3, \end{cases}$$
$$F_j x = \begin{cases} \frac{x^3-1+n}{n} & \text{if } 1 \leq x \leq 2, \\ \frac{n+1}{n} & \text{if } 2 < x \leq 3, \end{cases} \quad j = 2n\ (n \geq 1) \quad \text{and} \quad Sx = \begin{cases} 4x^6 - 3 & \text{if } 1 \leq x \leq 2, \\ 13 & \text{if } 2 < x \leq 3. \end{cases}$$

Note that $\partial C = \{1,3\}$. Clearly, for each $x \in C$ and $y \notin C$, there exists a point $z = 3 \in \partial C$ such that $d(x,z)+d(z,y) = d(x,y)$. Further, $SC \cap TC = [1,253] \cap [1,61] = [1,61] \supset \{1,3\} = \partial C$, $F_iC \cap C = [1,\frac{n+3}{n}] \cap [1,3] \subset SC$ and $F_jC \cap C = [1,\frac{n+7}{n}] \cap [1,3] \subset TC$.

Also,

$$T1 = 1 \in \partial C \quad \Rightarrow \quad F_i 1 = 1 \in C, \qquad S1 = 1 \in \partial C \quad \Rightarrow \quad F_j 1 = 1 \in C,$$

$$T\left(\sqrt[4]{\frac{3}{2}}\right) = 3 \in \partial C \quad \Rightarrow \quad F_i\left(\sqrt[4]{\frac{3}{2}}\right) = \frac{\sqrt{\frac{3}{2}} - 1}{n} + 1 \in C,$$

$$S\left(\sqrt[6]{\frac{3}{2}}\right) = 3 \in \partial C \quad \Rightarrow \quad F_j\left(\sqrt[6]{\frac{3}{2}}\right) = \frac{\sqrt{\frac{3}{2}} - 1}{n} + 1 \in C.$$

Moreover, if $x \in [1,2]$ and $y \in [2,3]$, then

$$d(F_ix, F_jy) = \frac{1}{n}\left|x^2 - 2\right|\varphi = \frac{|x^4 - 4|}{n|x^2 + 2|}\varphi = \frac{4|x^4 - 4|}{4n|x^2 + 2|}\varphi = \frac{1}{4n(x^2 + 2)}d(Tx, Sy).$$

Next, if $x, y \in (2,3]$, then

$$d(F_ix, F_jy) = 0 = \lambda \cdot d(Tx, Sy).$$

Finally, if $x, y \in [1,2]$, then

$$d(F_ix, F_jy) = \frac{1}{n}\left|x^2 - y^3\right|\varphi = \frac{|x^4 - y^6|}{n|x^2 + y^3|}\varphi = \frac{4|x^4 - y^6|}{4n|x^2 + y^3|}\varphi = \frac{1}{4n(x^2 + y^3)}d(Tx, Sy).$$

Therefore, condition (2.2) is satisfied if we choose $\lambda = \max\{\frac{1}{4n(x^2+2)}, \frac{1}{4n(x^2+y^3)}\} \in (0,1)$. Moreover, 1 is a point of coincidence as $T1 = F_i1$ as well as $S1 = F_j1$, whereas both the pairs (F_i, T) and (F_j, S) are weakly compatible as $TF_i1 = 1 = F_iT1$ and $SF_j1 = 1 = F_jS1$. Also, SC, TC, F_iC and F_jC are closed in X. Thus, all the conditions of Theorem 2.2 are satisfied and 1 is the unique common fixed point of F_i, F_j, S and T. One may note that 1 is also a point of coincidence for both the pairs (F_i, T) and (F_j, S).

Remark 2.1 Setting $F_i = F$ and $F_j = G$ in Theorem 2.2, we obtain the following result.

Corollary 2.1 *Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $F, G, S, T : C \to X$ are such that (F,G) is a pair of generalized $(T,S)_{M_1}$-contractive mappings of C into X, and

(I) $\partial C \subseteq SC \cap TC$, $FC \cap C \subseteq SC$, $GC \cap C \subseteq TC$,

(II) *$Tx \in \partial C$ implies that $Fx \in C$, $Sx \in \partial C$ implies that $Gx \in C$,*

(III) *SC and TC (or FC and GC) are closed in X.*

Then

(IV) *(F,T) has a point of coincidence,*

(V) *(G,S) has a point of coincidence.*

Moreover, if (F,T) and (G,S) are coincidentally commuting pairs, then F, G, S and T have a unique common fixed point.

Remark 2.2 1. Theorem 2.2 in [10] is a special case of Theorem 2.2 with $F_i = F_j = f$, $T = S = g$ and $\lambda \in (0, \sqrt{2} - 1)$.

2. Setting $F_i = F_j = f$ and $T = S = I_X$ (the identity mapping on X) in Theorem 2.2, we obtain the following result.

Corollary 2.2 *Let (X,d) be a complete cone metric space, and let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $f : C \to X$ satisfies the condition

$$d(fx,fy) \preceq \lambda u(x,y),$$

where

$$u(x,y) \in \left\{ d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy) + d(y,fx)}{2} \right\}$$

for all $x,y \in C$, $\lambda \in [0,1)$ and f has the additional property that for each $x \in \partial C$, $fx \in C$. Then f has a unique fixed point.

Remark 2.3 The following definition is a special case of Definition 2.1 when (X,d) is a metric space. But when (X,d) is a cone metric space, which is not a metric space, this is not true. Indeed, there may exist $x,y \in X$ such that the vectors $d(Tx,F_ix)$, $d(Sy,F_jy)$ and $\frac{d(Tx,F_ix)+d(Sy,F_jy)}{2}$ are incomparable. For the same reason Theorems 2.2 and 2.3 (given below) are incomparable.

Definition 2.2 Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X, and let $\{F_n\}_{n=1}^{\infty}, S, T : C \to X$ be non-self mappings. Denote, for $x,y \in C$,

$$M_2^{F_n,S,T} = \left\{ d(Tx,Sy), \frac{d(Tx,F_ix) + d(Sy,F_jy)}{2}, \frac{d(Tx,F_jy) + d(F_ix,Sy)}{2} \right\}, \tag{2.20}$$

where $i = 2n-1$, $j = 2n$ for some $n \in N$. Then (F_i,F_j) is called a pair of generalized $(T,S)_{M_2}$-contractive mappings of C into X if for some $\lambda \in [0,1)$ there exists $u(x,y) \in M_2^{F_n,S,T}$ such that for all $x,y \in C$ with $x \neq y$,

$$d(F_ix,F_jy) \preceq \lambda u(x,y). \tag{2.21}$$

Our next result is the following.

Theorem 2.3 *Let (X,d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $F_n, S, T : C \to X$ are such that (F_i, F_j) is a pair of generalized $(T,S)_{M_2}$-contractive mappings of C into X for all $i = 2n - 1$, $j = 2n$ ($n \in N$), and

(I) $\partial C \subseteq SC \cap TC$, $F_iC \cap C \subseteq SC$, $F_jC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $F_ix \in C$, $Sx \in \partial C$ *implies that* $F_jx \in C$,

(III) SC *and* TC (*or* F_iC *and* F_jC) *are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence*,

(V) (F_j, S) *has a point of coincidence*.

Moreover, if (F_i, T) and (F_j, S) are coincidentally commuting pairs, then F_n, S and T have a unique common fixed point.

The proof of this theorem is very similar to the proof of Theorem 2.2 and it is omitted.

Remark 2.4 Setting $F_i = F$ and $F_j = G$ in Theorem 2.3, we obtain the following result.

Corollary 2.3 *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $F, G, S, T : C \to X$ are such that (F, G) is a pair of generalized $(T,S)_{M_2}$-contractive mappings of C into X, and

(I) $\partial C \subseteq SC \cap TC$, $FC \cap C \subseteq SC$, $GC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $Fx \in C$, $Sx \in \partial C$ *implies that* $Gx \in C$,

(III) SC *and* TC (*or* FC *and* GC) *are closed in* X.

Then

(IV) (F, T) *has a point of coincidence*,

(V) (G, S) *has a point of coincidence*.

Moreover, if (F, T) and (G, S) are coincidentally commuting pairs, then F, G, S and T have a unique common fixed point.

We now list some corollaries of Theorems 2.2 and 2.3.

Corollary 2.4 *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F_n, S, T : C \to X$ be such that

$$d(F_ix, F_jy) \preceq \lambda d(Tx, Sy) \tag{2.22}$$

for some $\lambda \in [0,1)$ and for all $i = 2n - 1$, $j = 2n$ ($n \in N$), $x, y \in C$ with $x \neq y$.

Suppose, further, that F_n, S, T and C satisfy the following conditions:

(I) $\partial C \subseteq SC \cap TC$, $F_iC \cap C \subseteq SC$, $F_jC \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $F_ix \in C$, $Sx \in \partial C$ *implies that* $F_jx \in C$,

(III) SC *and* TC (*or* F_iC *and* F_jC) *are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence*,

(V) (F_j, S) *has a point of coincidence*.

Moreover, if (F_i, T) *and* (F_j, S) *are coincidentally commuting pairs, then* $\{F_n\}_{n=1}^{\infty}$, S *and* T *have a unique common fixed point*.

Corollary 2.5 *Let* (X, d) *be a complete cone metric space, let* C *be a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F_n, S, T : C \to X$ *be such that*

$$d(F_i x, F_j y) \preceq \lambda\big(d(Tx, F_i x) + d(Sy, F_j y)\big) \tag{2.23}$$

for some $\lambda \in [0, 1/2)$ *and for all* $i = 2n - 1$, $j = 2n$ $(n \in N)$, $x, y \in C$ *with* $x \neq y$.

Suppose, further, that F_n, S, T *and* C *satisfy the following conditions*:

(I) $\partial C \subseteq SC \cap TC$, $F_i C \cap C \subseteq SC$, $F_j C \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $F_i x \in C$, $Sx \in \partial C$ *implies that* $F_j x \in C$,

(III) SC *and* TC (*or* $F_i C$ *and* $F_j C$) *are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence*,

(V) (F_j, S) *has a point of coincidence*.

Moreover, if (F_i, T) *and* (F_j, S) *are coincidentally commuting pairs, then* $\{F_n\}_{n=1}^{\infty}$, S *and* T *have a unique common fixed point*.

Corollary 2.6 *Let* (X, d) *be a complete cone metric space, let* C *be a nonempty closed subset of* X *such that for each* $x \in C$ *and* $y \notin C$ *there exists a point* $z \in \partial C$ *such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $F_n, S, T : C \to X$ *be such that*

$$d(F_i x, F_j y) \preceq \lambda\big(d(Tx, F_j y) + d(F_i x, Sy)\big) \tag{2.24}$$

for some $\lambda \in [0, 1/2)$ *and for all* $i = 2n - 1$, $j = 2n$ $(n \in N)$, $x, y \in C$ *with* $x \neq y$.

Suppose, further, that F_n, S, T *and* C *satisfy the following conditions*:

(I) $\partial C \subseteq SC \cap TC$, $F_i C \cap C \subseteq SC$, $F_j C \cap C \subseteq TC$,

(II) $Tx \in \partial C$ *implies that* $F_i x \in C$, $Sx \in \partial C$ *implies that* $F_j x \in C$,

(III) SC *and* TC (*or* $F_i C$ *and* $F_j C$) *are closed in* X.

Then

(IV) (F_i, T) *has a point of coincidence*,

(V) (F_j, S) *has a point of coincidence*.

Moreover, if (F_i, T) *and* (F_j, S) *are coincidentally commuting pairs, then* $\{F_n\}_{n=1}^{\infty}$, S *and* T *have a unique common fixed point*.

Remark 2.5 Setting $F_i = F_j = f$ and $T = S = g$ in Corollaries 2.4-2.6, we obtain the following result.

Corollary 2.7 *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $f, g : C \to X$ be such that

$$d(fx,fy) \preceq \lambda d(gx,gy) \tag{2.25}$$

for some $\lambda \in [0,1)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) *$gx \in \partial C$ implies that $fx \in C$*,
(III) *gC is closed in X*.

Then there exists a coincidence point z of f, g in C. Moreover, if (f,g) are coincidentally commuting, then z is the unique common fixed point of f and g.

Corollary 2.8 *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $f, g : C \to X$ be such that

$$d(fx,fy) \preceq \lambda\big(d(fx,gx) + d(fy,gy)\big) \tag{2.26}$$

for some $\lambda \in [0,1/2)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,
(II) *$gx \in \partial C$ implies that $fx \in C$*,
(III) *gC is closed in X*.

Then there exists a coincidence point z of f, g in C. Moreover, if (f,g) are coincidentally commuting, then z is the unique common fixed point of f and g.

Corollary 2.9 *Let (X, d) be a complete cone metric space, let C be a nonempty closed subset of X such that for each $x \in C$ and $y \notin C$ there exists a point $z \in \partial C$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Let $f, g : C \to X$ be such that

$$d(fx,fy) \preceq \lambda\big(d(fx,gy) + d(fy,gx)\big) \tag{2.27}$$

for some $\lambda \in [0,1/2)$ and for all $x, y \in C$. Suppose, further, that f, g and C satisfy the following conditions:

(I) $\partial C \subseteq gC, fC \cap C \subseteq gC$,

(II) $gx \in \partial C$ *implies that* $fx \in C$,

(III) gC *is closed in* X.

Then there exists a coincidence point z *of* f, g *in* C. *Moreover, if* (f, g) *are coincidentally commuting, then* z *is the unique common fixed point of* f *and* g.

Remark 2.6 Corollaries 2.7-2.9 are the corresponding theorems of Abbas and Jungck from [2] in the case that f, g are non-self mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

1. Huang, LG, Zhang, X: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468-1476 (2007)
2. Abbas, M, Jungck, G: Common fixed point results for noncommuting mappings without continuity in cone metric spaces. J. Math. Anal. Appl. **341**(1), 416-420 (2008)
3. Abbas, M, Rhoades, BE: Fixed and periodic point results in cone metric spaces. Appl. Math. Lett. **22**, 511-515 (2009)
4. Berinde, V: Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces. Taiwan. J. Math. **14**(5), 1763-1776 (2010)
5. Huang, XJ, Zhu, CX, Wen, X: Common fixed point theorem for four non-self mappings in cone metric spaces. Fixed Point Theory Appl. **2010**, Article ID 983802 (2010). doi:10.1155/2010/983802
6. Huang, XJ, Zhu, CX, Wen, X: Common fixed point theorem for a family of non-self mappings in cone metric spaces. Hacet. J. Math. Stat. **41**(6), 853-865 (2012)
7. Huang, XJ, Zhu, CX, Wen, X: Fixed point theorems for expanding mappings in cone metric spaces. Math. Rep. **14**(2), 141-148 (2012)
8. Ilić, D, Rakocević, V: Common fixed points for maps on cone metric space. J. Math. Anal. Appl. **341**(2), 876-882 (2008)
9. Janković, S, Kadelburg, Z, Radenović, S: On cone metric spaces: a survey. Nonlinear Anal. **74**, 2591-2601 (2011)
10. Janković, S, Kadelburg, Z, Radenović, S, Rhoades, BE: Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces. Fixed Point Theory Appl. **2009**, Article ID 761086 (2009). doi:10.1155/2009/761086
11. Jungck, G, Radenovic, S, Radojević, S, Rakočević, V: Common fixed point theorems for weakly compatible pairs on cone metric spaces. Fixed Point Theory Appl. **2009**, Article ID 643840 (2009). doi:10.1155/2009/643840
12. Kadelburg, Z, Radenović, S, Rakočević, V: A note on the equivalence of some metric and cone metric fixed point results. Appl. Math. Lett. **24**, 370-374 (2011)
13. Radenović, S: A pair of non-self mappings in cone metric spaces. Kragujev. J. Math. **36**(2), 189-198 (2012)
14. Radenović, S, Rhoades, BE: Fixed point theorem for two non-self mappings in cone metric spaces. Comput. Math. Appl. **57**, 1701-1707 (2009)
15. Rezapour, S, Hamlbarani, R: Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. J. Math. Anal. Appl. **345**, 719-724 (2008)
16. Vetro, P: Common fixed points in cone metric spaces. Rend. Circ. Mat. Palermo **LVI**, 464-468 (2007)
17. Wardowski, D: Endpoints and fixed points of set-valued contractions in cone metric space. Nonlinear Anal. **71**, 512-516 (2009)
18. Assad, NA, Kirk, WA: Fixed point theorems for set valued mappings of contractive type. Pac. J. Math. **43**(3), 553-562 (1972)
19. Wong, YC, Ng, KF: Partially Ordered Topological Vector Spaces. Clarendon, Oxford (1973)
20. Hadzic, O, Gajic, L: Coincidence points for set-valued mappings in convex metric spaces. Univ. Novom. Sadu. Zb. Rad. Prirod. Mat. Fak. Ser. Mat. **16**(1), 13-25 (1986)
21. Imdad, M, Kumar, S: Rhoades-type fixed point theorems for a pair of non-self mappings. Comput. Math. Appl. **46**, 919-927 (2003)

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