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THE WEAK AND THE STRONG EQUIVALENCE RELATION AND THE ASYMPTOTIC INVERSION

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Abstract

In this paper we discuss the relationship between the weak and the strong asymptotic equivalence relation and the asymptotic inversion, for positive and measurable functions defined on a half-axis $[a,+\infty)$ $(a>0)$.

As the main results, we prove a certain characterizations of the functional class of all rapidly varying functions, as well as some other functional classes.

1 Introduction

A function $f:[a,+\infty)\mapsto(0,+\infty)$ $(a>0)$ is called *$\mathcal{O}$–regularly varying in the sense of Karamata* if it is measurable and

$$\overline{k}_f(\lambda):=\overline{\lim_{x\to+\infty}}\frac{f(\lambda x)}{f(x)}<+\infty\quad(\lambda>0). \tag{1}$$

Condition (1) is equivalent to the condition

$$\underline{k}_f(\lambda):=\underline{\lim_{x\to+\infty}}\frac{f(\lambda x)}{f(x)}>0\quad(\lambda>0). \tag{2}$$

$\overline{k}_f(\lambda)$ $(\lambda>0)$ is called the *index function* of f, and $\underline{k}_f(\lambda)$ $(\lambda>0)$ the *auxiliary index function* of f. ORV is the class of all $\mathcal{O}$–regularly varying functions defined on some interval $[a,+\infty)$.

A function $f\in ORV$ is called *regularly varying* in sense of Karamata if $\overline{k}_f(\lambda)=\lambda^\rho$ for all $\lambda>0$ and some $\rho\in\mathbb{R}$; then, ρ is the general index of variability of f. The class of all regularly varying functions is denoted RV. This class is the main object of the Karamata theory of regular variability (e.g. see [14]) and its applications (see also [1], [2] and [15]).

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A function $f \in RV$ is called *slowly varying* in the sense of Karamata (see e.g. [14]), if its general index of variability $\rho = 0$. This class of functions is denoted by SV (see [2] and [15]).

A measurable function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ belongs to the class PI if $\underline{k}_f\ (\lambda_0) > 1$ for some $\lambda_0 > 1$ ([5]).

A measurable function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ belongs to the class PI^* if there is a $\lambda_0 \geq 1$ such that

$$\underline{k}_f(\lambda) > 1, \quad \text{for all } \lambda > \lambda_0.$$

For $\lambda_0 = 1$ we obtain the class ARV (see [11]).

The class PI^* is a subclass of the class PI (see e.g.[5]) More information about these classes can be found in [7] and [12].

A function $f \in ARV$ is called *rapidly varying* in the sense of de Haan, with index ∞ (i.e. belonging to the class R_∞) if $\underline{k}_f(\lambda) = +\infty$ for all $\lambda > 1$ (see [2], [6] and [13]). The class PI^* contains as a proper subclass, the class of regularly varying functions whose Karamata index of variability ρ is positive, but it does not contain any element from the class of slowly varying Karamata functions.

Next, let

$$\mathcal{A} = \{f : [a, +\infty) \mapsto (0, +\infty)(a > 0) \mid f \text{ is nondecreasing and unbounded}\}.$$

Note that $\mathcal{A} \cap PI^* = \mathcal{A} \cap PI$. Next, let $\mathcal{A}^0$ be the set of all functions $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$. We notice that $\mathcal{A} \subsetneq \mathcal{A}^0$. If $f \in \mathcal{A}^0$, define $\{f\} = \{g \in \mathcal{A}^0 \,|\, f(x) \asymp g(x), x \to +\infty\}$, where $f(x) \asymp g(x)$, $x \to +\infty$, is the weak asymptotic equivalence relation defined by

$$0 < \varliminf_{x \to +\infty} \frac{f(x)}{g(x)} \leq \varlimsup_{x \to +\infty} \frac{f(x)}{g(x)} < +\infty$$

(see e.g. [2]).

For any function $f \in \mathcal{A}^0$ put $[f] = \{g \in \mathcal{A}^0 \mid f(x) \sim g(x), x \to +\infty\}$, where $f(x) \sim g(x)$, $x \to +\infty$, is the strong asymptotic equivalence relation defined by

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

For any $f \in \mathcal{A}$, $f^{\leftarrow}(x) = \inf\{y \geq a \mid f(y) > x\}$ $(x \geq f(a))$ is called the *generalized inverse* of f (see e.g. [2]).

If $f \in \mathcal{A}$ is continuous and strictly increasing, then $f^{\leftarrow}(x) = f^{-1}(x)$ for $x \geq f(a)$. Besides, $f^{\leftarrow} \in \mathcal{A}$ whenever $f \in \mathcal{A}$. For any right continuous function $g \in \mathcal{A}$ there is an $f \in \mathcal{A}$ $(f(x) = g^{\leftarrow}(x), x \geq g(a))$ such that $g = f^{\leftarrow}$.

Two arbitrary functions $f, g \in \mathcal{A}^0$ are called *mutually inversely asymptotic* (which is denoted by $f(x) \overset{*}{\sim} g(x)$ as $x \to +\infty$), if for every $\lambda > 1$, there is an $x_0 = x_0(\lambda) \geq a$ such that

$$f(x/\lambda) \leq g(x) \leq f(\lambda x),$$

for every $x \geq x_0$ (see e.g. [1], [2] and [11]).

From a result in [2] we get that for any functions $f, g \in \mathcal{A}$ we have $f(x) \overset{*}{\sim} g(x)$ as $x \to +\infty$ if and only if $f^{\leftarrow}(x) \sim g^{\leftarrow}(x)$ as $x \to +\infty$.

In the next proposition (see e.g. [1] or [2]) a result, which was an initial motivation for considering similar problems (e.g. see [8], [10],[11], [12]), is obtained. This result is the main motivation for this paper, too.

Proposition A. *Let $f, g \in \mathcal{A}^0$ and $f \in RV$, where $\rho > 0$ is the general index of variability of f. If $f(x) \sim g(x)$ $(x \to +\infty)$, then $f(x) \overset{*}{\sim} g(x)$ $(x \to +\infty)$.*

Remark 1. In [3] and [4], several modifications of this proposition are considered.

In [8] and [11] some results which expand Proposition A are proved, and they are contained in the following proposition.

Proposition B. *Let $f, g \in \mathcal{A}^0$ and $f \in ARV$. If $f(x) \sim g(x)$ $(x \to +\infty)$, then $f(x) \overset{*}{\sim} g(x)$ $(x \to +\infty)$.*

In [11] the following question is posed:

Q_1: *Is the class ARV the widest possible class for which Proposition B is satisfied?*

The answer to this question is affirmative in the case when the functions f and g in Proposition B are from the class $\mathcal{A}$ instead from the class $\mathcal{A}^0$ (see [8] and [11]).

Two functions $f, g \in \mathcal{A}^0$ are called *mutually inverse weak asymptotic* (denoted $f(x) \overset{*}{\asymp} g(x)$ as $x \to +\infty$) if there is a $\lambda_0 \geq 1$ such that for every $\lambda > \lambda_0$ there is an $x_0 = x_0(\lambda) > 0$ so that

$$f(x/\lambda) \leq g(x) \leq f(\lambda x),$$

for all $x \geq x_0$ (see e.g. [12]).

From a result in [12] it follows that for arbitrary functions $f, g \in \mathcal{A}$ we have $f(x) \overset{*}{\asymp} g(x)$ as $x \to +\infty$, if and only if $f^{\leftarrow}(x) \asymp g^{\leftarrow}(x)$ as $x \to +\infty$.

Next result (see [12]) is a modification of Proposition A, i.e. B.

Proposition C. *Let $f, g \in \mathcal{A}^0$ and $f \in PI^*$. If $f(x) \asymp g(x)$ $(x \to +\infty)$, then $f(x) \overset{*}{\asymp} g(x)$ $(x \to +\infty)$.*

In [12] the following question is posed:

Q_2: *Is the class PI^* the widest possible class for which the Proposition C is satisfied?*

The answer to this question is affirmative if we replace $\mathcal{A}^0$ with $\mathcal{A}$ (see [12]).

Remark 2. In [9] the affirmative answer to question Q_2 is given, in the case when we consider only strictly increasing and continuous functions from the class $\mathcal{A}$.

2 Main results

In the following propositions we shall give the affirmative answers to the questions Q_1 and Q_2, so we shall get some characterizations for the classes ARV and PI^*.

Proposition 1. *Let f and g be arbitrary measurable functions from the class $\mathcal{A}^0$. If $f(x)\overset{*}{\sim}g(x)$ $(x \to +\infty)$ whenever $f(x) \sim g(x)$ $(x \to +\infty)$, then $f \in ARV$. Every g satisfying the above condition also belongs to ARV.*

Proof. Let $f \in \mathcal{A}^0$ be an arbitrary measurable function such that $f(x)\overset{*}{\sim}g(x)$ $(x \to +\infty)$ whenever $f(x) \sim g(x)$ $(x \to +\infty)$, $g \in \mathcal{A}^0$ and is measurable.

Take $g = f$. Since $f(x) \sim f(x)$ as $x \to +\infty$ we find that $f(x)\overset{*}{\sim}f(x)$ $(x \to +\infty)$. Hence, for every $\lambda > 1$, there is an $x_0(\lambda) = x_0 \geq a$ such that $f(x/\lambda) \leq f(x) \leq f(\lambda x)$ $(x \geq x_0)$. Consequently, $\frac{f(\lambda x)}{f(x)} \geq 1$, and so $\underline{k}_f(\lambda) \geq 1$ $(\lambda > 1)$. Next, we shall prove that $\underline{k}_f(\lambda) > 1$ for every $\lambda > 1$.

Contrarily, assume that there is a $\lambda > 1$ such that $\underline{k}_f(\lambda) = 1$. Now, we distinguish between two cases.

1^0. There is an increasing and unbounded sequence (x_n), $x_n \geq a$ $(n \in \mathbb{N})$ such that $\frac{f(\lambda x_n)}{f(x_n)} = 1$ $(n \in \mathbb{N})$. If we define $g(x) = (1 + \frac{1}{x}) \cdot f(x)$ $(x \geq a)$, we find that $f(x) \sim g(x)$ $(x \to +\infty)$, so that for those λ we have that

$$f(x/\lambda) \leq \left(1 + \frac{1}{x}\right) \cdot f(x) \leq f(\lambda x),$$

for every $x \geq x_0(\lambda) = x_0 \geq a$.

Hence, for those x and λ we have $\frac{f(\lambda x)}{f(x)} \geq 1 + \frac{1}{x} > 1$, and this also holds if x equals to some element of the sequence x_n which is greater than (or equal) a. But this obviously contradicts to the assumption from the case 1^0.

2^0. There is an increasing and unbounded sequence (x_n) $(x_n \geq a, n \in \mathbb{N})$ such that $\lim_{n\to+\infty} \frac{f(\lambda x_n)}{f(x_n)} = 1$ and a sequence $a_n > 1$ $(n \in \mathbb{N})$, where $a_n = \frac{f(\lambda x_n)}{f(x_n)}$. Notice that in this case, we also have that $\underline{k}_f(\lambda) = 1$.

Next, define a function $u(x)$ $(x \geq a)$ as follows: $u(x_n) = a_n$ $(n \in \mathbb{N})$, $u(x)$ is linear and continuous on every interval $[x_{n-1}, x_n]$ $(n \in \mathbb{N})$, where $x_0 = a$ and $u(a) = a_1$. Then $\lim_{x\to+\infty} u(x) = 1$. If we define $g(x) = u^2(x) \cdot f(x)$ $(x \geq a)$ then $g \in [f]$, and for the considered λ we have $f(x/\lambda) \leq u^2(x) \cdot f(x) \leq f(\lambda x)$ $(x \geq x_0(\lambda) = x_0 \geq a)$. Hence, for those x and λ we find that $\frac{f(\lambda x)}{f(x)} \geq u^2(x)$, and this inequality is also true for values x which are equal to the elements of the sequence x_n, which are greater than (or equal) a. Finally, for the same λ and sufficiently large n it follows that $\frac{f(\lambda x_n)}{f(x_n)} \geq u^2(x_n) > u(x_n) = a_n$. But this obviously contradicts the assumption from the Case 2^0.

Therefore, we have shown that $\underline{k}_f(\lambda) > 1$ for every $\lambda > 1$, so that $f \in ARV$.

Next, suppose that $f \in ARV$ and $g \in \mathcal{A}^0$ is such that $f(x) \sim g(x)$ as $x \to +\infty$. Then $\underline{k}_g(\lambda) \geq \underline{k}_f(\lambda) > 1$ for every $\lambda > 1$, so that $g \in ARV$. This completes the proof.

Proposition 2. *Let f and g be arbitrary measurable functions from the class $\mathcal{A}^0$. If $f(x)\overset{*}{\asymp}g(x)$ $(x \to +\infty)$ whenever $f(x) \asymp g(x)$ $(x \to +\infty)$, then $f \in PI^*$. All g satisfying the above condition also belong to PI^*.*

Proof. Let f be an arbitrary measurable function from the class $\mathcal{A}^0$ which satisfies the condition above. We shall prove that $f \in PI^*$. For $x \geq a$, define $g(x) = 2 \cdot f(x)$. Then $g \in \{f\}$, so there is a $\lambda_0 \geq 1$ such that for every $\lambda > \lambda_0$ we have $f(x/\lambda) \leq 2 \cdot f(x) \leq f(\lambda x)$ $(x \geq x_0(\lambda) = x_0 \geq a)$. For those λ and x we obtain $\frac{f(\lambda x)}{f(x)} \geq 2$, so that $\underline{k}_f(\lambda) = \underline{\lim}_{x\to+\infty} \frac{f(\lambda x)}{f(x)} \geq 2 > 1$. Thus, for every $\lambda > \lambda_0 \geq 1$ we have $\underline{k}_f(\lambda) > 1$ and hence $f \in PI^*$.

Next, let $f \in PI^* \cap \mathcal{A}^0$ be an arbitrary function, and g be arbitrary measurable function from the class $\mathcal{A}^0$ with the property $g \in \{f\}$. Then $g(x) = h(x) \cdot f(x)$ for $x \geq a$, where $h(x)$ $(x \geq a)$ is a measurable function, and $0 < 1/M \leq h(x) \leq M < +\infty$ for some $M > 1$ and all sufficiently large x. Consequently, we have that $\underline{k}_g(\lambda) \geq \dfrac{1}{M^2} \cdot \underline{k}_f(\lambda)$ for all $\lambda > \lambda_0 \geq 1$. Since $f \in PI^*$, by a result from [12] we find that $\lim\limits_{\lambda\to+\infty} \underline{k}_f(\lambda) = +\infty$. Therefore, for all sufficiently large x (and for all $\lambda > \lambda_1 \geq \lambda_0 \geq 1$) we have $\underline{k}_f(\lambda) > M^2$. This gives $\underline{k}_g(\lambda) > 1$ for all $\lambda > \lambda_1 \geq \lambda_0 \geq 1$. Hence, $g \in PI^*$. This completes the proof.

Further, we consider two more modifications of Proposition A, and for them we discuss the appropriate claims related to Propositions 1 and 2. In this way, we obtain some characterizations of the classes PI^* and R_∞.

Proposition 3. *Let $f, g \in \mathcal{A}^0$ and $f \in PI^*$. If $f(x) \sim g(x)$ $(x \to +\infty)$, then $f(x)\overset{*}{\asymp}g(x)$ $(x \to +\infty)$.*

Proof. The statement of this proposition is a direct corollary of Proposition C ([12]).

In the next proposition we prove that PI^* is the widest class of functions f, for which the previous proposition remains true.

Proposition 4. *Let f and g be arbitrary measurable functions from the class $\mathcal{A}^0$, and let $f(x)\overset{*}{\asymp}g(x)$ $(x \to +\infty)$ whenever $f(x) \sim g(x)$ $(x \to +\infty)$. Then $f \in PI^*$. All g satisfying the above condition also belong to PI^*.*

Proof. The proof of this claim is very similar to the corresponding proof of Proposition 1. The only difference is as follows: in Proposition 1 an arbitrary $\lambda > 1$ is taken, while here any $\lambda > \lambda_0 \geq 1$ for an arbitrary $\lambda_0 \geq 1$, is taken.

In the next two propositions we give a characterization of the class R_∞, the well-known de Haan's class of rapidly varying functions (see e.g. [13], [2], [6] and [10]).

Proposition 5. *Let f and g be arbitrary functions from the class $\mathcal{A}^0$ and $f \in R_\infty$. If $f(x) \asymp g(x)$ $(x \to +\infty)$, then $f(x) \overset{*}{\sim} g(x)$ $(x \to +\infty)$.*

Proof. First, note that $\frac{1}{M} \leq \frac{g(x)}{f(x)} \leq M$, for some $M > 1$ and every $x \geq x_1(M) = x_1 \geq a$. Further, for any $\lambda > 1$ we have that $\frac{f(x)}{f(\lambda x)} = c_f(\lambda, x) > 0$ $(x \geq a)$, and $\lim_{x\to+\infty} c_f(\lambda, x) = 0$. For those λ and $x \geq x_1$ we find that $g(x) \leq M \cdot f(x) = M \cdot c_f(\lambda, x) \cdot f(\lambda x)$. Assuming that $M < +\infty$, then for every $\lambda > 1$ we have that $g(x) \leq f(\lambda x)$ $(x \geq x_1^*(\lambda) = x_1^* \geq a)$, where $x_1^* = \max\{x_1, x_1'\}$ and $x_1' = x_1'(\lambda) \geq a$. Hence, $c_f(\lambda, x) \cdot M \leq 1$ for every $x \geq x_1'$.

Therefore, $g(x) \geq \frac{1}{M} \cdot f(x)$ for $x \geq x_1$. Since $\lim_{x\to+\infty} \frac{f(x)}{f(x/\lambda)} = +\infty$ for every $\lambda > 1$, we have $\frac{f(x)}{f(x/\lambda)} = d_f(\lambda, x) > 0$ for $x \geq a$, and $\lim_{x\to+\infty} d_f(\lambda, x) = +\infty$, for every $\lambda > 1$. Hence, for every $\lambda > 1$ we obtain $d_f(\lambda, x) \cdot \frac{1}{M} \geq 1$, for every $x \geq x_2' = x_2'(\lambda) \geq a$. In other words, for every $\lambda > 1$ we have that $g(x) \geq \frac{1}{M} \cdot f(x) = \frac{1}{M} \cdot d_f(\lambda, x) \cdot f(x/\lambda) \geq f(x/\lambda)$ for every $x \geq x_2^*(\lambda) = x_2^* \geq a$, where $x_2^* = \max\{x_1, x_2'\}$. Finally, for every $\lambda > 1$ we obtain that $f(x/\lambda) \leq g(x) \leq f(\lambda x)$ for every $x \geq x_0 = x_0(\lambda) = \max\{x_1^*, x_2^*\} \geq a$. This means that $f(x) \overset{*}{\sim} g(x)$ as $x \to +\infty$.

Proposition 6. *Let f and g be arbitrary measurable functions from the class $\mathcal{A}^0$. If $f(x) \overset{*}{\sim} g(x)$ $(x \to +\infty)$ whenever $f(x) \asymp g(x)$ $(x \to +\infty)$, then $f \in R_\infty$. All g satisfying the condition above also belong to R_∞.*

Proof. The prof of this proposition is mostly similar to the proof of Proposition 2, but some parts of these proofs differ. Hence, we shall give the entire proof.

Let f be an arbitrary measurable function from the class $\mathcal{A}^0$ and let α be an arbitrary positive number. Next, let $g(x) = \alpha \cdot f(x)$ $(x \geq a)$. Then $g \in \{f\}$ and we have that $f(x) \overset{*}{\sim} g(x)$ as $x \to +\infty$. Hence, for any $\lambda > 1$ we have that $f(x/\lambda) \leq g(x) \leq f(\lambda x)$, for every $x \geq x_0(\lambda) = x_0 \geq a$. For those λ and x we find that $f(x/\lambda) \leq \alpha \cdot f(x) \leq f(\lambda x)$.

Now, (for the same λ and x) we find that $\alpha \leq \frac{f(\lambda x)}{f(x)}$. Hence, for any $\lambda > 1$ we have that $\underline{k}_f(\lambda) \geq \alpha$, where $\alpha > 0$ is arbitrary. So, for any $\lambda > 1$, we have that $\lim_{\alpha\to+\infty} \underline{k}_f(\lambda) \geq \lim_{\alpha\to+\infty} \alpha = +\infty$, thus for these λ we have $\underline{k}_f(\lambda) = \overline{k}_f(\lambda) = +\infty$. Therefore, for any $\lambda > 1$, we obtain that $\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty$, i.e. we have $f \in R_\infty$.

Next, let $f \in R_\infty$ and $g \in \{f\}$, where $g \in \mathcal{A}^0$ and is measurable. As in the proof of Proposition 2 we have that $\underline{k}_g(\lambda) \geq \dfrac{1}{M^2} \cdot \underline{k}_f(\lambda) = +\infty$ for every $\lambda > 1$ and arbitrary large number $M > 1$. But this gives that $g \in R_\infty$. This completes the proof.

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